Embedding of the Non-Commutative CP(1) Model as a Gauge Theory

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In the present research we have evaluated the constraint structure of the CP(1) model in non-commutative (NC) space. Although most theories have first class constraints and are classed as gauge theories, the CP(1) model in non-commutative space is not a gauge theory, due to the appearance of second class constraints. The NC CP(1) model is converted to a gauge theory by using the BFT method by introducing some auxiliary fields, which in turn converts the second class constraints into first class ones. Finally we investigate the partition function of this model and show that it is ready to quantize in the usual way.

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I. INTRODUCTION

Non-commutative (NC) theories have turned into a hotbed issue of research activities after its connection to low energy string physics was elucidated by Seiberg and Witten [1, 2]. NC, specifically, is focused on the open string boundaries, attached to D-branes [3].

The appropriate NC field theory is constructed from NC analogue fields \( \hat{\psi} \) of the fields \( \psi \) with the replacement of ordinary products of fields \( \psi \varphi \), by the \(*\)-product \( \hat{\psi} \hat{\varphi} \),

\[
\hat{\psi}(x)\hat{\varphi}(x) = e^{\frac{i}{4} \theta_{\rho\sigma} \partial_\rho \varphi(x+\sigma) \partial_\sigma \varphi(x+\varepsilon)} \left|_{\sigma=\varepsilon=0} \right. = \hat{\psi}(x)\varphi(x) + \frac{i}{2} \theta^{\rho\sigma} \partial_\rho \psi(x) \partial_\sigma \varphi(x) + O(\theta^3).
\]

(1)

The hatted variables are NC degrees of freedom, and \( \theta^{\rho\sigma} \) is a real constant antisymmetric tensor [1]. The NC spacetime follows from the definition

\[
[x^\rho, x^\sigma]_* = i\theta^{\rho\sigma}.
\]

(2)

It should be noted that the effects of spacetime noncommutativity have been accounted for by the introduction of the \(*\)-product for the gauge theories that have been explained by the Seiberg-Witten map [1], which plays a crucial role in the connection of \( \varphi(x) \) to \( \varphi(x) \).

Canonical quantization of constrained systems is a powerful and rigorous way to quantize a classical model. In this method the systems are divided into three groups: first
class constraints, second class constraints, and mixed constraint systems. Physical systems with constraints were systematically developed by Dirac [4].

First class constraints are fully gauge invariant and the quantization methods work in a standard way, but in the other two groups the systems are not gauge invariant due to the second class constraint system, so we need to use the Dirac bracket instead of the Poisson bracket to proceed with the quantization of the physical system.

Ghosh analyzed the constraints using the Dirac bracket and defined the Poincare algebra and the equation of motion and gave an analysis of the Bogomolny bound for this model [5, 6]. However in this Dirac quantization scheme, we have some difficulties to find a canonical conjugate pair, due to the field operator ordering ambiguity. To circumvent such problems Batalin, Fradkin, and Tyutin (BFT) invented a scheme which converts the second class constraints into first class ones by introducing some auxiliary fields and embedding the phase space in a extended phase space. In this way we can convert the second class constraints to the first class one and make a gauge theory [7, 8]. In the present study we have used the BFT method that has not been used in Ghosh [5, 6] and the CP(1) model that was not evaluated in Monemzadeh et al. [9].

In Section II, first we will introduce the gauge invariant action CP(1) model in ordinary spacetime, and subsequently we will obtain the NC action for this model, and by using the energy-momentum tensor we will obtain the canonical Hamiltonian and the constraint structure. In Section III, the BFT method will be applied by the introduction of some auxiliary fields for the conversion of second class constraints to first class ones. In addition, we will also obtain the embedded Hamiltonians and constraints in the extended phase space. Finally we will show that the partition function of this model and NC CP(1) is ready to quantize in the usual way. Section IV is devoted to our conclusion.

II. NON-COMMUTATIVE (NC) CP(1) MODEL

The CP(1) model in ordinary spacetime is described by the gauge invariant action below [5, 6]:

\[ S = \int \left[ ((D^\mu \phi)^* D_\mu \phi + \Lambda (\phi^* \phi - 1)) \right] d^3 x, \]

where \( D_\mu \phi = (\partial_\mu - iA_\mu)\phi \) defines the covariant derivative, the multiplier \( \Lambda \) enforces the CP(1) constraint, and \( A_\mu \) is defined by \( A_\mu = -i\phi^* \partial_\mu \phi \). Let us enter the Non-Commutative spacetime in the CP(1) model. The first step is to generalize the scalar gauge theory Eq. (3) to its NC version, keeping in mind that the latter must be \(*\) - gauge invariant. The NC action is [5, 6]

\[ \hat{S} = \int (\hat{D}^\mu \hat{\phi})^* \hat{D}_\mu \hat{\phi} d^3 x = \int (\hat{D}^\mu \hat{\phi})^* \hat{D}_\mu \hat{\phi} d^3 x, \]

where the NC covariant derivative is defined as [5, 6]

\[ \hat{D}_\mu \hat{\phi} = \partial_\mu \hat{\phi} - i \hat{A}_\mu \hat{\phi}. \]
Now, we exploit the Seiberg-Witten map in order to revert back to the ordinary spacetime degrees of freedom. There is a relation between NC and ordinary spacetime counterparts of the fields. The lowest nontrivial orders of $\theta$ are $[5, 6]$

\[
\hat{A}_\mu = A_\mu + \theta^{\rho\sigma} A_\rho (\partial_{\sigma} A_\mu - \frac{1}{2} \partial_{\mu} A_{\sigma}),
\]  
\[
\hat{\phi} = \phi - \frac{1}{2} \theta^{\alpha\sigma} A_\rho \partial_{\sigma} \phi.
\]

The hatted variables on the left are the NC degrees of freedom and gauge transformation parameter. The higher order terms than $O(\theta^2)$ in $\theta$ were kept out. The NC action Eq. (4) was converted to being gauge invariant by applying the Seiberg-Witten map [1, 10]. Thus the NC modified action Eq. (4) becomes $[5, 6]$

\[
\hat{S} = \int d^3 x [(D^\mu \phi)^* D_\mu \phi + \frac{1}{2} \theta^{\alpha\beta} \{ F_{\alpha\beta}((D_\beta \phi)^* D^\mu \phi + (D^\mu \phi)^* D_\beta \phi) - \frac{1}{2} F_{\alpha\beta} (D^\mu \phi)^* D_\mu \phi \}],
\]

where $F_{\alpha\beta}$ is the Electromagnetic Field Tensor, which is also expressible as follow:

\[
F_{\alpha\beta} = D_\alpha A_\beta - D_\beta A_\alpha = D_\alpha (-i \phi^* D_\beta \phi) - D_\beta (-i \phi^* D_\alpha \phi)
\]  
\[
\equiv -i [D_\alpha \phi^* D_\beta \phi - D_\beta \phi^* D_\alpha \phi].
\]

Remember that so far we have introduced the CP(1) target space in the NC spacetime setup. Let’s assume the constraint to be identical to the ordinary spacetime one $[5, 6]$: $\phi^* \phi = 1.$

By defining the canonical momenta and the assumptions of $\theta^0 = 0 ; i = 1, 2$ (we shall mostly be interested in the space-space NC $\theta^0 = 0$ in this work. This might be expected because when time is NC, it is difficult to construct any sensible Hamiltonian formalism for the theory. Indeed, if time is NC, it is hard to say that one has a Hamiltonian at one instant of time, as every instant of time is related to another instant of time due to the infinite nonlocality of the $\ast$- product (1) [16]. Accordingly when $\theta^0$ does not vanish we could not write a Hamiltonian therefore we could not use the BFT method for this theory.) and $\theta^{12} = \theta^{21}$ the following constraints can be obtained:

\[
\phi \pi - \phi^* \pi^* \approx 0,
\]
\[
\phi \pi + \phi^* \pi^* \approx 0,
\]

where $\pi$ are the canonical momenta conjugate to the complex scalar fields $\phi$ given by

\[
\pi = (1 + C) D^0 \phi^* - i \theta \varepsilon^{ij} (D^0 \phi^* D^j \phi) D^i \phi^*,
\]
\[
\pi^* = (1 + C) D^0 \phi + i \theta \varepsilon^{ij} (D^0 \phi D^j \phi^*) D^i \phi,
\]

and $\pi^*$ are the complex conjugate of $\pi$ [5]. Now we have three constraints “using the Poisson brackets”. One of them is the first class constraint $\langle \Phi \rangle$ while the others are the second class
constraints ($\chi_1$ and $\chi_2$):
\[
\Phi \equiv \phi \pi - \phi^* \pi^* \approx 0, \quad (12)
\]
\[
\chi_1 \equiv \phi^* \phi - 1 \approx 0, \quad (13)
\]
\[
\chi_2 \equiv \phi \pi + \phi^* \pi^* \approx 0.
\]

It is clear that we have a mixed constraint system in which its constraint structure contains both first and second class constraints.

By using the energy-momentum tensor for the NC CP(1) model, we consider $\mu = 0$, $\nu = 0$ and we obtain the canonical Hamiltonian as follows [5, 6]:
\[
T_{00} = H_c = (\pi^* \pi + D^k \phi^* D^k \phi)(1 + C) + i\theta \varepsilon^{ij} (\pi^* D^i \phi^*)(\pi D^j \phi), \quad (14)
\]
where $C \equiv -\frac{1}{4} \theta \varepsilon^{ij} F_{ij}$. It is now straightforward to check the following equation [10]:
\[
\{\Phi, H_c\} = 0. \quad (15)
\]

The consistency condition of the primary constraint (i.e., $\Phi = \{\chi, H\}$) leads to the secondary constraint expressions
\[
\{\chi_1, H\} = (\phi \pi + \phi^* \pi^*)[(1 + C) + i\theta \varepsilon^{ij} D^i \phi^* D^j \phi], \quad (16)
\]
and
\[
\{\chi_2, H\} = \pi \pi^* [2(1 + C) + 2i\theta \varepsilon^{ij} D^i \phi^* D^j \phi]. \quad (17)
\]

As can be seen, Eq. (16) is not independent of the previous constraints, so the only secondary constraint is derived from Eq. (17) as $\pi = 0$. So by redefining the constraints the new first class constraints are
\[
\phi \pi + \phi^* \pi^* \approx 0, \quad (18)
\]
\[
\phi \pi - \phi^* \pi^* \approx 0,
\]
and the two second class constraints are
\[
\phi \phi^* - 1 \approx 0, \quad (19)
\]
\[
\pi \approx 0.
\]

It is clear that we have a mixed constraint system and we can apply the BFT method to it and convert the second class constraints to first class and make a full gauge theory.

**III. APPLYING THE BFT METHOD**

In this section we apply the BFT method to this model and make a gauge theory by introducing some auxiliary fields and embedding the phase space into an extended phase
space and converting the second class constraints to first class ones. To view more details and a brief review of this approach one can refer to some references [7–9, 11]. The NC CP(1) model of Eqs. (18–19) described a mixed constraint system. Since the Poisson brackets of the first class constraints and \( H_c \) vanish, so the first class constraints are not perturbed when we embed the second class constraints [12]. By using the second class constraints and the Poisson brackets of the constraints we can make a symplectic matrix:

\[
\Delta_{\alpha\beta} = \begin{pmatrix}
0 & \phi^* \\
-\phi^* & 0
\end{pmatrix}.
\] (20)

In a local space we can use the finite order BFT method [7], in this case we can choose the arbitrary parameters of the BFT approach as

\[
\omega_{\alpha\beta} = \Delta^T_{\alpha\beta} = -\Delta_{\alpha\beta};
\] (21)

\( \omega_{\alpha\beta} \) and \( \Delta_{\alpha\beta} \) are the inversion of \( \omega^{\alpha\beta} \) and \( \Delta^{\alpha\beta} \), respectively. However, the basic equation of the BFT can be written as

\[
\Delta - \chi^T \Delta \chi = 0.
\] (22)

It is easy to see that \( \chi = 1 \) satisfies the above equation. It can be seen clearly, that for \( n \geq 1 \), \( B_{\alpha(2)} \) vanishes.

In order to convert the gauge noninvariant NC CP(1) to first class constraints, we use two new auxiliary fields \( \eta^1 \) and \( \eta^2 \) as follows:

\[
\tau_{1(1)} = \eta^1,
\]

\[
\tau_{2(1)} = \eta^2,
\] (23)

so the new set of first class constraints are

\[
\tau_1 \equiv (\phi^* \phi - 1) + \eta^1 \approx 0,
\]

\[
\tau_2 \equiv \pi + \eta^2 \approx 0.
\] (24)

Now the function \( G_{\alpha(0)} \) is the generator of \( \hat{H}^{(n+1)} \). We can obtain them for \( n = 0 \) as follows:

\[
G_{1(0)} = (\phi \pi + \phi^* \pi^*)[(1 + C) + i \theta \varepsilon^{ij} D_i \phi^* D^j \phi],
\]

\[
G_{2(0)} = 0,
\] (25)

and for \( n = 1 \), \( G_{\alpha(1)} \) would be

\[
G_{1(1)} = -2 \phi \eta^2[(1 + C) + i \theta \varepsilon^{ij} D_i \phi^* D^j \phi],
\]

\[
G_{2(1)} = \frac{\pi \eta^2}{\phi^*}[(1 + C) + i \theta \varepsilon^{ij} D_i \phi^* D^j \phi],
\] (26)
and for $n = 2$ we have

$$G^{(2)}_1 = \frac{1}{2\phi^*} \eta^2 \eta^1[(1 + C) + i\theta \varepsilon^{ij} D^i \phi^* D^j \phi],$$

$$G^{(2)}_2 = -\frac{1}{\phi^*} \eta^2 \eta^2[(1 + C) + i\theta \varepsilon^{ij} D^i \phi^* D^j \phi].$$

(27)

It is straightforward to check that the other functions for $n \geq 3$, $G^{(n)}_\alpha$ vanish. We can find the correction terms of the Hamiltonian as follows:

$$\tilde{H}^{(1)} = -\frac{1}{\phi^*} \eta^2[\phi \pi + \phi^* \pi^*][(1 + C) + i\theta \varepsilon^{ij} D^i \phi^* D^j \phi],$$

(28)

$$\tilde{H}^{(2)} = \left(\frac{\phi}{\phi^*} \eta^2 \eta^2 + \frac{\pi}{2(\phi^*)^2} \eta^2 \eta^1\right)[(1 + C) + i\theta \varepsilon^{ij} D^i \phi^* D^j \phi],$$

(29)

$$\tilde{H}^{(3)} = -\frac{1}{2(\phi^*)^2} \eta^2 \eta^2 \eta^1[(1 + C) + i\theta \varepsilon^{ij} D^i \phi^* D^j \phi],$$

(30)

and finally the Hamiltonian in the extended phase space is

$$\tilde{H} = H_c + \tilde{H}^{(1)} + \tilde{H}^{(2)} + \tilde{H}^{(3)}.$$

(31)

We converted the second class constraints into first class and obtained a first class Hamiltonian for this model by Eq. (31), so now we have a gauge theory.

Finally, the partition function is given by the Faddev formula as follows [13]:

$$Z = \int D\phi^* D\phi D\eta^1 D\eta^2 \prod_{i,j=1}^{3} \delta(\tau_i) \delta(\Gamma_j) \det |\{\tau_i, \Gamma_j\}| e^{iS},$$

(32)

where $S$ is defined as

$$S = \int d^3 x \tilde{L} = \int d^3 x (\pi \dot{\phi} + \pi^* \dot{\phi}^* - \tilde{H}).$$

(33)

For deriving Eq. (33) we have used the Legendre transformation $\tilde{L} = p \dot{q} - \tilde{H}$, where $\dot{q} = \frac{\partial \tilde{H}}{\partial p}$ and $(q, p)$ are dynamical variables of the extended phase space. $\Gamma_j$ are gauge fixing conditions and are chosen in such a way that the determinate functional measure is nonvanishing. Usually $\Gamma_j$ may be assumed to be independent of the momenta, as these are considered in Faddeev-Popov type gauge conditions [11, 14, 15]. The procedure is completed and the model is ready to quantize in this formal approach.

IV. CONCLUSION

Gauge theories fall under the class of constrained system that divides the constraints into two groups: first and second class constraints. First class constraints are generators of
gauge transformations and are present in a gauge theory while, systems possessing second class constraints are not gauge theories.

In this paper we noted that the NC CP(1) model, due to the second class constraint, is not a gauge theory. We wanted to do the quantization of the theory, but quantization of a second class system is not trivial and is more difficult, as compared with first class ones. We thus used the BFT method and converted the NC CP(1) model into a gauge theory by introducing some auxiliary fields, which in turn converts the second class constraints to first class ones. We obtained the new set of first class constraints, as can be seen in Eq. (24); we also obtained the generator $G^{(n)}_\alpha$ which is mentioned in Eq. (25)–(27) and the Hamiltonian in the extended phase space appears in Eqs. (28)–(30); at the end we investigated the partition function of this model and show that it is ready to quantize in a usual way.

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References