Application of Polynomial $su(2)$ Algebra to a Physical System with Symmetric Rosen-Morse Potential

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(Received November 28, 2013; Revised December 14, 2014)

A novel polynomial $su(2)$ algebra is constructed in a physical system with the symmetric Rosen-Morse potential. Meanwhile, its specific representations are presented spontaneously. The polynomial $su(2)$ algebra can be used as an algebraic technique to solve for the eigenvalues and eigenfunctions of the Hamiltonian with the symmetric Rosen-Morse potential (SRM). The output results from the mentioned algebraic approach allows us to explore the pair of raising and lowering operators $J_\pm$ of the polynomial $su(2)$ algebra as a pair of shift operators of our Hamiltonian. In addition, the usual $su(2)$ algebra is obtained naturally from the polynomial $su(2)$ algebra we built, also it reveals that the physical system with the SRM potential has a dynamical $su(2)$ symmetry.

DOI: 10.6122/CJP.20150126A PACS numbers: 03.65.Fd, 03.65-w, 02.20Sv

I. INTRODUCTION

Polynomial algebras are a particular type of non-linear deformation of Lie algebras [1] and have been applied to many quantum mechanical models [2–4]. Different realizations of polynomial algebras have also been widely explored in [5–7]. In particular, polynomial deformed $su(2)$ and $su(1,1)$ algebras have been extensively studied by several authors as a new and powerful algebraic technique for solving for the energy eigenvalues and eigenfunctions of some solvable quantum mechanical systems [8–10]. They mainly focus on the quadratic and cubic algebras, which are regarded as special cases of polynomial $su(2)$ and $su(1,1)$ algebras [11, 12]. The cubic algebra was first considered by Higgs and Leeman in dealing with the harmonic oscillator and the Kepler problem on a two-dimensional sphere [13, 14], while the quadratic algebra was first analyzed by Sklyanin in conjunction with the quantum group [15]. So far, the cubic algebra has appeared in various areas of study including the identical particle symmetry in two dimensions, the two-body Calogero-Sutherland model, multi-photon processes, quantum dot problem, and others [16–20]. All of the above mentioned have revealed that a major advantage of the algebraic approach is to avoid solving the complicated second-order differential Schrödinger equation. For this intention a number of studies have been undertaken to construct different polynomial algebras for some solvable physical models [21–23].

The aim of this paper is to construct a novel polynomial $su(2)$ algebra associated with

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the symmetric Rosen-Morse (SRM) potential by extending Ref. [24, 25]. Then we will use the polynomial algebra and its specific representations to calculate one of the main results of this paper, that is the exact eigenfunctions and energy eigenvalues of the Hamiltonian with the SRM potential.

This work is organized as follows: in Section II we first give a brief review of the polynomial \( su(2) \) algebra and its representations, next we construct an appropriate polynomial \( su(2) \) algebra for the SRM potential, and last we show its corresponding specific representations. In Section III, based on those results, the energy spectrum and eigenfunctions of the Hamiltonian with the SRM potential are obtained by an algebraic technique. The usual \( su(2) \) algebra is obtained from the polynomial \( su(2) \) algebra naturally in Section IV. Some conclusions are summarized in the last section.

II. POLYNOMIAL \( su(2) \) ALGEBRA REALIZED IN A PHYSICAL SYSTEM WITH THE SRM POTENTIAL

In [26], Bonatsos, Daskaloyannis, and Kolokotronis proposed a deformed \( su(2) \) algebra, denoted by \( su_\Phi(2) \), which has representations similar to those of the usual \( su(2) \). In their deformation, the three generators \( \{ \hat{J}_\pm, \hat{J}_0 \} \) of the algebra obey the commutation relations

\[
[\hat{J}_0, \hat{J}_\pm] = \pm \hat{J}_\pm, \quad [\hat{J}_+, \hat{J}_-] = \Phi(\hat{J}_0(\hat{J}_0 + 1)) - \Phi(\hat{J}_0(\hat{J}_0 - 1)),
\]

where the structure function \( \Phi(x) \) is an increasing function of \( x \) defined for \( x \geq -\frac{1}{4} \). If \( x \) is a Hermitian operator, it is operator-valued. The Casimir operator for \( su_\Phi(1, 1) \) is

\[
\hat{J}_0^2 = \hat{J}_\pm \hat{J}_\mp + \Phi \left( \hat{J}_0(\hat{J}_0 - 1) \right).
\]

On the basis \( |j, n\rangle \) that diagonalizes \( \hat{J}_0^2 \) and \( \hat{J}_0 \) simultaneously such that

\[
\hat{J}_0^2 |j, n\rangle = \Phi(j(j + 1)) |j, n\rangle, \quad \hat{J}_0 |j, n\rangle = (n - j) |j, n\rangle,
\]

the operators \( \hat{J}_\pm \) satisfy the relations

\[
\hat{J}_\pm |j, n\rangle = \sqrt{\Phi(j(j + 1)) - \Phi((n - j)(n - j \pm 1))} |j, n \pm 1\rangle
\]

with \( n = 0, 1, 2, \ldots, 2j \) and \( j > 0 \).

Now, we consider a physical system with a particle of mass \( M \) moving in the SRM potential, which takes the form [27, 28]

\[
V(x) = -\frac{V_0}{\cosh^2 \left( \frac{x}{\alpha} \right)},
\]
where $V_0$ and $a$ are positive constants. By a change of variable $\xi = \frac{x}{a}$, introducing two parameters $\epsilon = \frac{h^2}{2Ma^2}$ and $V_0 = \epsilon j(j+1)$, then the corresponding Hamiltonian can be expressed as

$$\hat{H} = \epsilon \left( \frac{\hat{p}_x^2}{\hbar^2} \frac{j(j+1)}{\cosh^2 \xi} \right).$$

As is well known, the physical system with the SRM potential can be exactly solved by dealing with the Schrödinger equation directly; the eigenfunctions of energy are represented by hypergeometric functions [27, 28]. In the following, we will build the polynomial su(2) algebra and its specific representation through utilizing Ref. [24]'s method. Based on these results we derive energy values and corresponding energy eigenfunctions by an algebraic approach. Firstly, following the strategy of Ref. [24], is to search for the polynomial su(2) algebra. Let us begin with the definitions

$$\hat{X} = \sinh \xi, \quad \frac{\hat{p}_x}{\hbar} = \frac{1}{2} \{ \cosh \xi, \hat{p}_x \},$$

where the symbol $\{,\}$ represents the anti-commutator. Using the fundamental commutation relations $[\sinh \xi, \frac{\hat{p}_x}{\hbar}] = \cosh \xi$ and $[\cosh \xi, \frac{\hat{p}_x}{\hbar}] = \sinh \xi$, we obtain

$$\left[ \hat{X}, \frac{\hat{p}_x}{\hbar} \right] = 1 + \hat{X}^2, \quad [\hat{H}, \hat{X}] = 2\epsilon \frac{\hat{p}_x}{\hbar}, \quad [\hat{H}, \frac{\hat{p}_x}{\hbar}] = -\frac{\epsilon}{2} \left\{ \frac{4\hat{p}_x}{\hbar} + \hat{X} \left( \frac{4\hat{H}}{\epsilon} + 1 \right) \right\}. \quad (9)$$

Besides, the Hamiltonian can be turned into the following identity

$$4j(j+1) + \frac{4\hat{H}}{\epsilon} = 4\frac{\hat{p}_x^2}{\hbar^2} - 2 \left( 3\hat{X} \frac{\hat{p}_x}{\hbar} + \frac{\hat{p}_x}{\hbar} \hat{X} \right) - \hat{X}^2 \left( 3 + \frac{4\hat{H}}{\epsilon} \right). \quad (10)$$

The last two equations in (9) can be expressed in matrix form as follows:

$$\hat{H} \left( \begin{array}{c} \hat{X} \\ \frac{\hat{p}_x}{\hbar} \end{array} \right) = \left( \begin{array}{c} \hat{X} \\ \frac{\hat{p}_x}{\hbar} \end{array} \right) \hat{G},$$

where $\hat{G}$ is the $2 \times 2$ matrix, $\hat{G} = \left( \begin{array}{cc} \hat{H} & -2\hat{H} - \frac{j}{\epsilon} \\ 2\epsilon & \hat{H} - 2\epsilon \end{array} \right)$. To diagonalize $\hat{G}$, solving the characteristic equation $\det(\hat{G} - \lambda I) = 0$ we may derive the two eigenvalues of $\hat{G}$ as follows:

$$\lambda_{\pm} = \hat{H} - \epsilon \pm 2\sqrt{-\epsilon \hat{H}} = -\left( \sqrt{\epsilon} \mp \sqrt{-\hat{H}} \right)^2. \quad (12)$$

Hence, the diagonalized $\hat{G}$ can be written in the form $\hat{G} = \hat{S} \hat{\Lambda} \hat{S}^{-1}$, where the diagonal matrix $\Lambda = \left( \begin{array}{cc} \lambda_+ & 0 \\ 0 & \lambda_- \end{array} \right)$ and the transformation matrix $S = \left( \begin{array}{cc} \epsilon + 2\sqrt{-\epsilon \hat{H}} & \epsilon - 2\sqrt{-\epsilon \hat{H}} \\ 2\epsilon & 2\epsilon \end{array} \right)$. 
Let $F(\hat{H})$ be a real function of $\hat{H}$, holomorphic in the neighborhood of zero. From (11) we have the operator equation

$$F(\hat{H}) \left( \hat{X}, \frac{\hat{P}}{i\hbar} \right) = \left( \hat{X}, \frac{\hat{P}}{i\hbar} \right) SF(\Lambda) S^{-1}. \quad (13)$$

In order to find the polynomial $su(2)$ algebra, for convenience we take two special cases of (13) for calculation:

(i). When $F(\hat{H}) = \sqrt{-\hat{H}}$, then $\hat{S}F(\Lambda)\hat{S}^{-1} = \frac{1}{4\sqrt{-\hat{H}}} \begin{pmatrix} -4\hat{H} - 2\epsilon & 4\hat{H} + \epsilon \\ -4 & 2\epsilon - 4\hat{H} \end{pmatrix}$. Hence from (13) we acquire

$$\sqrt{-\hat{H}} \hat{X} = -\left\{ \hat{X} \left( 2\hat{H} + \epsilon \right) + 2\epsilon \frac{\hat{P}}{i\hbar} \right\} \frac{1}{2\sqrt{-\hat{H}}},$$

$$\sqrt{-\hat{H}} \frac{\hat{P}}{i\hbar} = \left\{ \hat{X} \left( 4\hat{H} + \epsilon \right) + 2\epsilon \frac{\hat{P}}{i\hbar} \left( \epsilon - 2\hat{H} \right) \right\} \frac{1}{4\sqrt{-\hat{H}}}. \quad (14)$$

Note that (14) can still be expressed in terms of two commutation relations as

$$[\sqrt{-\hat{H}}, \hat{X}] = -\left( \hat{X} + 2\frac{\hat{P}}{i\hbar} \right) \frac{\epsilon}{2\sqrt{-\hat{H}}},$$

$$[\sqrt{-\hat{H}}, \frac{\hat{P}}{i\hbar}] = \left\{ \hat{X} \left( 2\epsilon \hat{H} + \epsilon \right) + 2\epsilon \hat{P} \right\} \frac{1}{4\sqrt{-\hat{H}}}. \quad (15)$$

(ii). Choose $F(\hat{H}) = \frac{1}{\sqrt{-\hat{H}}}$, then $\hat{S}F(\Lambda)\hat{S}^{-1} = \frac{1}{4\sqrt{-\hat{H}(\hat{H} + \epsilon)}} \begin{pmatrix} 4\hat{H} - 2\epsilon & 4\hat{H} + \epsilon \\ -4\epsilon & 4\hat{H} + 2\epsilon \end{pmatrix}$. Thus, (13) is turned into

$$\frac{1}{\sqrt{-\hat{H}}} \hat{X} = -\left\{ \hat{X} \left( \epsilon - 2\hat{H} \right) + 2\epsilon \frac{\hat{P}}{i\hbar} \right\} \frac{1}{2\sqrt{-\hat{H}}} \left( \hat{H} + \epsilon \right),$$

$$\frac{1}{\sqrt{-\hat{H}}} \frac{\hat{P}}{i\hbar} = \left\{ \frac{1}{2} \hat{X} \left( \epsilon + 4\hat{H} \right) + \epsilon \frac{\hat{P}}{i\hbar} \left( 2\epsilon + \hat{H} \right) \right\} \frac{1}{2\sqrt{-\hat{H}}} \left( \hat{H} + \epsilon \right). \quad (16)$$

By means of a detailed investigation into (15), we define

$$\begin{cases} 
\hat{j}_0 = \sqrt{-\frac{\hbar}{\epsilon}}, \\
\hat{j}_- = \gamma \left[ \hat{X}(2\hat{j}_0 + 1) + 2\frac{\hat{P}}{i\hbar} \right], \\
\hat{j}_+ = (\hat{j}_-)^\dagger = \gamma \left[ (2\hat{j}_0 + 1)\hat{X} - 2\frac{\hat{P}}{i\hbar} \right] = \gamma \left[ \hat{X}(2\hat{j}_0 - 1) - 2\frac{\hat{P}}{i\hbar} \right] \frac{\hat{j}_{0+1}}{\hat{j}_0},
\end{cases} \quad (17)$$
where $\gamma$ is a changeable parameter.

From Equations (9) and (15), we immediately get two important commutation relations:

$$
\left[ J_0, J_\pm \right] = \pm J_\pm .
$$

Hence, we have defined the operator $\hat{J}_0, \hat{J}_\pm$ in (17) and successfully embedded the operators $\hat{X}, \hat{P}, \hat{H}$ into the polynomial $su(2)$ algebra. To make clear the physical significance of the operators $\hat{J}_\pm$, from Equations (18) one can obtain the commutation relations

$$
\left[ \hat{H}, \hat{J}_- \right] = -\hat{J}_- g(\hat{H}), \quad \left[ \hat{H}, \hat{J}_+ \right] = g(\hat{H}) \hat{J}_+ ,
$$

where $g(\hat{H}) = \epsilon \left( 1 - 2\sqrt{-\hat{H}/\epsilon} \right)$. Based on this, the physical meanings of $\hat{J}_\pm$ are very clear, namely, they are just the raising and lowering operators for the energy levels. Thus, we can see that the relations (18) are not only the cornerstone of the polynomial $su(2)$ algebra but also play a critical role in solving for the energy eigenfunctions.

By utilizing Equations (10), (14), and (16), we obtain a rather simple expression:

$$
\hat{J}_\pm \hat{J}_\mp = 4\gamma^2 \left[ j(j+1) - \hat{J}_0 \left( \hat{J}_0 \mp 1 \right) \right] \frac{\hat{J}_0 + \frac{1}{2} \mp \frac{1}{2}}{\hat{J}_0 - \frac{1}{2} \mp \frac{1}{2}} .
$$

On the one hand, comparing (20) with the Casimir operator (2) we require

$$
\Phi \left( \hat{J}_0(\hat{J}_0 \mp 1) \right) = \hat{J}^2 - 4\gamma^2 \left[ j(j+1) - \hat{J}_0 \left( \hat{J}_0 \mp 1 \right) \right] \frac{\hat{J}_0 + \frac{1}{2} \mp \frac{1}{2}}{\hat{J}_0 - \frac{1}{2} \mp \frac{1}{2}} ,
$$

thereby obtaining the expression of the structure function $\Phi(x)$:

$$
\Phi(x) = \Phi \left( j(j+1) \right) - 4\gamma^2 \left[ j(j+1) - x \right] \frac{\sqrt{4x+1} + 1}{\sqrt{4x+1} - 1} .
$$

On the other hand, from (20) and (21) we also obtain another important commutation relation:

$$
\left[ \hat{J}_+, \hat{J}_- \right] = \Phi(\hat{J}_0(\hat{J}_0 + 1)) - \Phi(\hat{J}_0(\hat{J}_0 - 1)) ,
$$

where $\Phi(\hat{J}_0(\hat{J}_0 + 1)) - \Phi(\hat{J}_0(\hat{J}_0 - 1)) = 4\gamma^2 \left[ 1 + 2\hat{J}_0 + \frac{j(j+1)}{\hat{J}_0(\hat{J}_0 - 1)} \right]$.

The results (18) and (23) exhibit that the set of operators $\{ \hat{J}_\pm, \hat{J}_0 \}$ defined by (17) establishes a polynomial $su(2)$ algebra. Inserting (8) into (17), namely, we have the polynomial $su(2)$ algebra in differential form for the SRM potential (6) or the Hamiltonian (7):

$$
\begin{cases}
\hat{J}_0 = \sqrt{-\frac{\hat{H}}{\epsilon}} , \\
\hat{J}_\pm = 2\gamma \left[ \sinh \xi \hat{J}_0 \mp \cosh \xi \left( \frac{\beta_k}{\hbar} \right) \right] \frac{\hat{J}_0 + \frac{1}{2} \mp \frac{1}{2}}{\hat{J}_0} .
\end{cases}
$$
In addition, inserting (22) into (2), the representations for the polynomial algebra (24) are specified as

\[ \hat{J}_2^2|j, n\rangle = \Phi(j(j + 1))|j, n\rangle, \]  
\[ \hat{J}_0|j, n\rangle = (n - j)|j, n\rangle, \]  
\[ \hat{J}_\pm|j, n\rangle = 2\gamma\beta_n^\pm|j, n \pm 1\rangle, \]  

where \( \beta_n^\pm = \sqrt{\frac{(n + j + \frac{1}{2})(2j - n + \frac{1}{2})(n - j - \frac{1}{2})}{j - n + j + \frac{1}{2}}} \).

The last two formulas are very useful for determining the eigenvalues and corresponding eigenfunctions of the Hamiltonian with the SRM potential.

III. DETERMINATION OF EIGEN-ENERGIES AND EIGENFUNCTIONS OF THE SYSTEM

From (24) we may express (7) with respect to the generators of the polynomial \( su(2) \) algebra as

\[ \hat{H} = -\epsilon \hat{J}_2^2. \]  

With the aid of the representation equation (26), the energy eigenequation can be obtained:

\[ \hat{H}|j, n\rangle = E_n|j, n\rangle, \]  

where the energy eigenvalues are

\[ E_n = -\epsilon(n - j)^2 = -\frac{\hbar^2}{2Ma^2}(n - j)^2, \quad n = 0, 1, 2, \ldots, 2j \]  

and the corresponding energy eigenstates are \( |j, n\rangle \).

The ground energy eigenfunction can be obtained by applying (24) to the lowering operation equation (27), which shows \( \langle \xi|\hat{J}_-|j, 0\rangle = 0 \) when \( n = 0 \), namely,

\[ \left( j \sinh \xi + \cosh \xi \frac{d}{d\xi} \right) \psi_0(\xi) = 0. \]  

Its normalized solution is

\[ \psi_0(\xi) = C_0 \cosh^{-j} \xi, \]  

where \( C_0 = \sqrt{\Gamma(j + \frac{1}{2})/\Gamma(j)} \) is a normalization constant, which can be determined by the normalization condition.

Substituting (24) into the raising operation equation (27), we get

\[ \frac{j - n + 1}{j - n} \left( (n - j) \sinh \xi + \cosh \xi \frac{d}{d\xi} \right) \psi_n(\xi) = \beta_n^+ \psi_{n+1}(\xi). \]
When choosing $n = 0$, from (33) the first excited wavefunction is given as $\psi_1(\xi) = C_1 \cosh^{-j} \xi \sinh \xi$ with $C_1 = -\sqrt{\frac{2\Gamma(j+\frac{1}{2})}{\Gamma\left(j^{\frac{1}{2}}\right)\Gamma(1-j)}}$. In principle, from (33) all excited energy eigenfunctions can be generated step by step from $\psi_0(\xi)$, but in fact it is a rather tedious process. Nevertheless, from the above we know that $\psi_0(\xi)$ and $\psi_1(\xi)$ are, respectively, even and odd functions of $\xi$; they are eigenfunctions of parity with eigenvalues $+1$ and $-1$. To achieve a general expression for all excited energy eigenfunctions, we have to consider raising $n$ by two. Hence starting from $\langle \xi | J_\mp^2 | j, n \rangle$ and using (27), we find

$$\psi_{n+2}(\xi) = -\frac{2(j-n-2)(j-n-1)}{(j-n)\beta_n^+\beta_{n+1}^+} \left[ \frac{(n+1)(2j-n)}{2(j-n-1)} - (j-n)\sin^2 \xi + \frac{1}{2} \sinh \frac{d}{d\xi} \right] \psi_n(\xi),$$

where we have used the result

$$J_\pm^2 = 4\gamma^2 \left[ \hat{X}^2 \left( 2\hat{J}_0 - 1 \right) - 2\hat{X} \frac{\hat{P}}{i\hbar} \right] \left( \hat{J}_0 + 1 \right) + \left( \hat{J}_0 - j \right) \left( \hat{J}_0 + j + 1 \right) \frac{\hat{J}_0 + 2}{\hat{J}_0} \psi_j(\xi).$$

We distinguish (34) into two cases for convenience of discussion. For $n = 2N + \frac{1}{2} \mp \frac{1}{2}$ ($N = 0, 1, 2, \cdots$) where minus and plus in $\mp$ stand for even and odd numbers, respectively, then (34) is written as

$$\psi_{2N+\frac{5}{2}+\frac{1}{2}}(\xi) = -\frac{2(j-2N-\frac{5}{2} \pm \frac{1}{2})(j-2N-\frac{3}{2} \pm \frac{1}{2})}{(j-2N-\frac{1}{2} \pm \frac{1}{2})\beta_{2N+\frac{5}{2}+\frac{1}{2}}^+\beta_{2N+\frac{3}{2}+\frac{1}{2}}^+} \left[ \frac{(2N+\frac{3}{2} \pm \frac{1}{2})(2j-2N-\frac{1}{2} \pm \frac{1}{2})}{2(j-2N-\frac{1}{2} \pm \frac{1}{2})} \right] \psi_{2N+\frac{3}{2}+\frac{1}{2}}(\xi).$$

By repeated application of (36) to $\psi_0(\xi)$ or $\psi_1(\xi)$, we can generate all the even wavefunctions $\psi_{2N}(\xi)$ or all the odd wavefunctions $\psi_{2N+1}(\xi)$ if we write

$$\psi_{2N+\frac{3}{2}+\frac{1}{2}}(\xi) = C_{\frac{3}{2}+\frac{1}{2}} \sinh^{\frac{1}{2}+\frac{1}{2}} \xi \cosh^{-j} \xi \phi_{2N+\frac{3}{2}+\frac{1}{2}}(\xi),$$

and change the variable to $\eta = -\sinh^2 \xi$, then (36) provides the recurrence relation

$$\phi_{2N+\frac{3}{2}+\frac{1}{2}}(\eta) = \Delta_{2N+\frac{3}{2}+\frac{1}{2}} \bar{D}_{2N+\frac{3}{2}+\frac{1}{2}}(\eta) \phi_{2N+\frac{3}{2}+\frac{1}{2}}(\eta), \quad (N = 0, 1, 2, \cdots)$$

where

$$\Delta_{2N+\frac{3}{2}+\frac{1}{2}} = -\frac{2(2N + 2 \mp 1)(j - N - \frac{1}{2} \pm \frac{1}{2})(j - 2N - \frac{5}{2} \pm \frac{1}{2})}{(j - 2N - \frac{1}{2} \pm \frac{1}{2})\beta_{2N+\frac{3}{2}+\frac{1}{2}}^+\beta_{2N+\frac{3}{2}+\frac{1}{2}}^+} \psi_{2N+\frac{3}{2}+\frac{1}{2}}(\eta).$$
Finally, collecting the above results (37), (41), (44), and (45) we obtain all the eigenfunctions of the symmetric Rosen-Morse potential by this algebraic technique. Obviously they are coincident with those obtained from the solution of the Schrödinger equation [27].

$$\psi_{2N+\frac{1}{2}+\frac{1}{2}}(\xi) = (−1)^{N+\frac{1}{2}+\frac{1}{2}} \sqrt{\frac{2^{1+1}(j - 2N - j + \frac{1}{2} \pm \frac{1}{2})\Gamma(N + 1 \pm \frac{1}{2})\Gamma(j - N + \frac{1}{2})}{N! [\Gamma(\frac{1}{2})]^{2}\Gamma(j - N + \frac{1}{2} \pm \frac{1}{2})}} \times 2F_1(-N, N - j + \frac{1}{2} \pm \frac{1}{2}; 1 \mp \frac{1}{2}; 1 \mp \frac{1}{2}, -\sin^2\xi \cosh^{-j}\xi (\sinh\xi)^{\frac{1}{2}+\frac{1}{2}}}. \tag{46}$$

for $N = 0, 1, 2, \cdots, j$. Thus, we have obtained the energy spectrum (30) and wavefunctions (46) for the symmetric Rosen-Morse potential by this algebraic technique. Obviously they are coincident with those obtained from the solution of the Schrödinger equation [27].
IV. FROM POLYNOMIAL $su(2)$ ALGEBRA TO USUAL $su(2)$ ALGEBRA

Comparing the commutation relations (18) and (23) of the polynomial $su(2)$ algebra with the commutation relations of a usual $su(2)$ algebra, we select

$$\hat{I}_0 = \hat{J}_0, \quad \hat{I}_+ = \hat{J}_+ f(\hat{J}_0), \quad \hat{I}_- = f(\hat{J}_0)\hat{J}_-.$$  \hspace{1cm} (47)

where $\hat{I}_\pm$ and $\hat{I}_0$ express the generators of the usual $su(2)$ algebra, and $f(x)$ is an undetermined function. Substituting (47) into $\hat{I}_2 = \hat{I}_0 (\hat{J}_0 \mp 1) + \hat{I}_\pm \hat{I}_\mp$ which is the Casimir of the usual $su(2)$ algebra, and noting that (20), we get

$$f(\hat{J}_0 \mp \frac{1}{2}) = \sqrt{\frac{\hat{I}_2 - \hat{J}_0 (\hat{J}_0 \mp 1)}{\hat{J}_0 \hat{J}_\mp}} = \frac{1}{2\gamma} \sqrt{\frac{(\hat{J}_0 - \frac{1}{2} \mp \frac{1}{2})}{j(j + 1) - \hat{J}_0 (\hat{J}_0 \mp 1)}} \frac{(\hat{J}_0 + \frac{1}{2} \pm \frac{1}{2})}{(\hat{J}_0 + \frac{1}{2} \mp \frac{1}{2})}. \hspace{1cm} (48)$$

If we replace $\hat{I}_2$ by $j(j + 1)$, then the above equation is simplified as

$$f(\hat{J}_0) = \frac{1}{2\gamma} \sqrt{\frac{\hat{J}_0}{\hat{J}_0 + 1}}. \hspace{1cm} (49)$$

From above calculation we see that $f(x)$ depends on the physical system we discussed, and different physical models correspond to different $f(x)$. With the aid of Equations (18), (23), and (49) we can easily prove that the operators $\hat{I}_{\pm,0}$ satisfy the commutation relations

$$[\hat{I}_0, \hat{I}_\pm] = \pm \hat{I}_\pm, \quad [\hat{I}_+, \hat{I}_-] = 2\hat{I}_0,$$

which corresponds to the usual $su(2)$ algebra. Therefore, collecting the above equations (24), (47), and (49), the usual $su(2)$ for the SRM potential is given as

$$\begin{align*}
\hat{I}_0 &= \sqrt{-\frac{\hat{I}_2}{\hat{J}_0}} \\
\hat{I}_\pm &= \left[ \sinh \xi \hat{I}_0 \mp \cosh \xi \left( \frac{\hat{p}_\xi}{\hbar} \right) \right] \sqrt{\frac{\hat{I}_0 + 1}{\hat{I}_0}}.
\end{align*} \hspace{1cm} (51)$$

In addition, from (47) and (49) we can derive that the Casimir operator $\hat{I}_2$ of the usual $su(2)$ algebra is $\hat{I}_2 = j(j + 1) = \frac{\hat{J}_0}{2}$. This tells us that the Casimir operator is a constant, and implies that the physical system with the SRM potential has a dynamical $su(2)$ symmetry.

V. CONCLUSIONS

Let us now quickly summarize the work. We have successfully constructed a polynomial $su(2)$ algebra (24) in differential form for a physical system with the SRM potential,
and derived the specific expressions (26)–(27) of its representations. Based on those results, the exact eigenfunctions and eigenvalues of the Hamiltonian with the SRM potential have been calculated by employing an algebraic technique. Moreover, the usual $su(2)$ is also obtained from the polynomial $su(2)$ algebra naturally. Also it shows that the physical system with the SRM potential has a dynamical $su(2)$ symmetry. During solving for the above results, we have shown that the Hamiltonian with the SRM potential can be expressed by $J_0^2$, which is related to the energy eigenvalues. However, the operators $J_\pm$ act as a pair of shift operators on the energy eigenvalues. Due to the different actions of these operators in the representations of the polynomial $su(2)$ algebra, the energy eigenvalues and corresponding eigenfunctions can be easily obtained. This also exhibits that the algebraic technique provides a simply way to find exact eigenvalues and eigenfunctions of such a class of the Hamiltonians associated with hypergeometric function (HG) or confluent hypergeometric function (CHG). Therefore, we believe that it may be used to deal with other solvable potentials connecting with the HG or CHG.

References


