The Control and Synchronization of a Rotational Relativistic Chaotic System via States Recovery

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This paper investigates the control and synchronization of a rotational relativistic chaotic system for which it is assumed that only one state is available. By constructing the proper observer, the unavailable states can be recovered. Some novel criteria for control or synchronization are proposed via a single input. Numerical simulations are given to demonstrate the robustness and efficiency of the proposed approach.

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I. INTRODUCTION

The study of chaos is one of the most active and prolific areas in nonlinear science owing to interesting properties, such as being extremely sensitive to tiny variations of initial conditions, having broad spectrums for its Fourier transform, and fractal properties of the motion in the phase plane. The control and synchronization of chaotic systems are currently under major consideration by numerous researchers due to their potential applications in biological systems, chemical reactions, power converters, information processing, secure communications, etc. Different techniques and methods have been proposed to achieve chaos control and synchronization of various chaotic systems, such as the OGY method [1], feedback approach [2], optimization method [3], adaptive control [4], impulsive control [5], $H_\infty$ approach [6], active control [7], sliding mode control [8], backstepping method [9], etc. However, most of the above mentioned works on chaos control and synchronization are under the hypotheses that all the state variables of the chaotic system are available. But, in practical situations some of the systems’ states cannot be exactly measured a priori. For example, in some system only the output is available, and the output may be a one state variable. Therefore, the investigation of the control and synchronization of chaotic systems with only one state available becomes an important topic.

Since Carmeli first proposed the theory of rotational relativistic mechanics in 1985 [10], the rotational relativistic system has been an active research topic [11–13]. In paper [14], the authors presented the nonlinear electromechanical coupling relative rotation system. The mechanism and conditions of the system parameters for chaotic motions were investigated rigorously based on the Silnikov method. So far as we know, less attention has been paid to the issue of control and synchronization of the new rotational relativistic

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chaotic system.

Motivated by the above discussion, in this paper we study the control and synchronization of the new rotational relativistic chaotic system, in which it is assumed that only one state is available. By constructing the proper observer, the unavailable states can be recovered. Some novel criteria for control or synchronization are proposed via a single input. Numerical simulations are given to demonstrate the robustness and efficiency of the proposed approach.

The layout of the rest of this paper is organized as follows: A brief description of the new rotational relativistic chaotic system is introduced in Section II. Section III investigates the design of the observer of the new chaotic system. The control and synchronization schemes of the new chaotic system are respectively proposed in Section IV and Section V. Section VI includes several numerical examples to demonstrate the effectiveness of the proposed approach; finally, some conclusions are shown in Section VII.

II. SYSTEM DESCRIPTION

The new chaotic system is described as [14]:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= a_1 x_1 + b_1 x_2 + c_1 x_3 + d_1 x_4 + l_1 x_1^2 + l_2 x_1^3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= a_2 x_1 + b_2 x_2 + c_2 x_3 + d_2 x_4,
\end{align*}
\]

where \(x_1, x_2, x_3, x_4\) are the state variables, \(a_i, b_i, c_i, d_i, l_i, i = 1, 2\) are system parameters. When \(b_1 = -0.22, c_1 = 0.11, d_1 = 0.05, a_2 = -10.28, b_2 = 0.05, c_2 = -0.11, d_2 = -0.05\) the system (1) is chaotic [14]. The chaotic attractor with \(a_1 = 2\) and \(x_1(0) = 1, x_2(0) = 2, x_3(0) = 3, x_4(0) = 1\) is shown in Fig. 1.

FIG. 1: The chaotic attractor of system (1). (a) Chaotic attractor in \((x_1; x_2; x_3)\) space; (b) Chaotic attractor in \((x_2; x_3; x_4)\) space.
Throughout this paper we make the following assumptions.

**Assumption 1.** Suppose the parameters satisfy $c_1 > 0$, $d_1 > 0$, and $c_2 < 0$.

As it is known that when $b_1 = -0.22$, $c_1 = 0.11$, $d_1 = 0.05$, $a_2 = -10.28$, $b_2 = 0.05$, $c_2 = -0.11$, $d_2 = -0.05$ system (1) is chaotic, so Assumption 1 is reasonable.

**Assumption 2.** Suppose the available state variable of system (1) is $x_1$. It is well known that not all of the state variables are available for some chaotic systems. For example, in some systems only the output is available, and the output may be a state variable like $x_1$ in system (1). Thus, we can assume the available state variable of system (1) is $x_1$. On the other hand, in the control or synchronization of chaotic systems most of the controllers are designed by using all of the systems' state variables. In order to construct a suitable controller we use the estimated values instead of the unavailable variables. Thus, the unavailable variables must be estimated for designing the proper controller. For this end in the next section we give the observer of system (1).

### III. THE OBSERVER OF THE NEW CHAOTIC SYSTEM

In order to recover the states $x_2$, $x_3$, and $x_4$ of system (1), we introduce the following observer:

$$
\begin{align*}
\dot{x}_2 &= w + (k + b_1)x_1, \\
\dot{w} &= -k\dot{x}_2 + c_1\dot{x}_3 + d_1\dot{x}_4 + a_1x_1 + l_1x_1^2 + l_2x_1^3, \\
\dot{x}_3 &= \dot{x}_4, \\
\dot{x}_4 &= a_2x_1 + b_2\dot{x}_2 + c_2\dot{x}_3 + d_2\dot{x}_4,
\end{align*}
$$

(2)

where $\dot{x}_2$, $\dot{x}_3$, and $\dot{x}_4$ are the estimated values of $x_2$, $x_3$, and $x_4$, respectively, and $k > 0$ is a constant to be designed later.

In the following, we will show that system (2) is the observer of system (1).

**Theorem 1.** If there exist $k$ and $l$ such that

$$
\begin{align*}
k &> -\frac{(c_1 + b_2)l^2}{4c_2l}, \\
k &> \frac{k(c_2l(d_2 + l) - \frac{1}{4}l^2(l + d_2)^2) + \frac{1}{2}|c_1 + b_2l|((\frac{1}{2}(d_2 + l)|c_1 + b_2l| - \frac{1}{2}|b_2 + d_1||l(l + d_2)| - \frac{1}{2}|b_2 + d_1||l(l + d_2)| - \frac{1}{2}c_2l|b_2 + d_1|)}{d_2 + l} > 0,
\end{align*}
$$

(3)

then system (2) is a observer of system (1), where $l > 0$ is a constant to be designed later.

**Proof.** Let us define $e_{2x} = \dot{x}_2 - x_2$, $e_{3x} = \dot{x}_3 - x_3$, $e_{4x} = \dot{x}_4 - x_4$, then from systems
(1) and (2) we have
\[ \dot{e}_{2x} = \dot{x}_2 - \dot{x}_2 = \dot{w} + (k + b_1)x_2 - (a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 + l_1x_1^2 + l_2x_1^3) \]
\[ = -k\dot{x}_2 + c_1\dot{x}_3 + d_1\dot{x}_4 + a_1x_1 + l_1x_1^2 + l_2x_1^2 + (k + b_1)x_2 \]
\[ = -ke_{2x} + c_1e_{3x} + d_1e_{4x}, \]
\[ \dot{e}_{3x} = \dot{x}_3 - \dot{x}_3 = e_{4x}, \]
\[ \dot{e}_{4x} = \dot{x}_4 - \dot{x}_4 = (a_2x_1 + b_2\dot{x}_2 + c_2\dot{x}_3 + d_2\dot{x}_4) - (a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4) \]
\[ = b_2e_{2x} + c_2e_{3x} + d_2e_{4x}. \]

Thus we get the error dynamical system as follows:
\[
\begin{cases}
\dot{e}_{2x} = -ke_{2x} + c_1e_{3x} + d_1e_{4x}, \\
\dot{e}_{3x} = \dot{x}_3 - \dot{x}_3 = e_{4x}, \\
\dot{e}_{4x} = b_2e_{2x} + c_2e_{3x} + d_2e_{4x}.
\end{cases}
\tag{4}
\]

For system (4) we can prove that
\[
\lim_{t \to \infty} e_{2x} = \lim_{t \to \infty} e_{3x} = \lim_{t \to \infty} e_{4x} = 0.
\]

In fact, we choose the Lyapunov candidate as:
\[
V = \frac{1}{2}e_{2x}^2 + \frac{1}{2}(-c_2)e_{3x}^2 + \frac{1}{2}(l_0e_{3x} + e_{4x})^2,
\]
where \( c_2 < 0 \).

Its derivative along the trajectories of system (4) is
\[
\dot{V} = e_{2x}\dot{e}_{2x} + (-c_2)e_{3x}\dot{e}_{3x} + (l_0e_{3x} + e_{4x})(l_0\dot{e}_{3x} + \dot{e}_{4x}) \\
= e_{2x}(-ke_{2x} + c_1e_{3x} + d_1e_{4x}) + (-c_2)e_{3x}e_{4x} \\
+ (l_0e_{3x} + e_{4x})(l_0\dot{e}_{3x} + \dot{e}_{4x}) + b_2e_{2x} + c_2e_{3x} + d_2e_{4x} \\
= -ke_{2x}^2 + c_2e_{3x}^2 + (d_2 + l)e_{4x}^2 + (c_1 + b_2)e_{2x}e_{3x} \\
+ (b_2 + d_1)e_{2x}e_{4x} + l(d_2 + l)e_{3x}e_{4x} \\
\leq -ke_{2x}^2 + c_2e_{3x}^2 + (d_2 + l)e_{4x}^2 + |c_1 + b_2||e_{2x}e_{3x}| \\
+ |b_2 + d_1||e_{2x}e_{4x}| + |l(d_2 + l)||e_{3x}e_{4x}| \\
= -(e_{2x}, e_{3x}, e_{4x})Q(e_{2x}, e_{3x}, e_{4x})^T,
\tag{5}
\]
where
\[
Q = \begin{pmatrix}
    k & -\frac{1}{2}|c_1 + b_2| & -\frac{1}{2}|b_2 + d_1| \\
    -\frac{1}{2}|c_1 + b_2| & -c_2l & -\frac{1}{2}|l(d_2 + d_2)| \\
    -\frac{1}{2}|b_2 + d_1| & -\frac{1}{2}|l(d_2 + d_2)| & -(d_2 + l)
\end{pmatrix}.
\]

In view of the inequality (3), one can easily know that the three principal minor determinants of \( Q \) are greater than 0, which implies that \( Q \) is a positive definite matrix. Thus the three eigenvalues of \( Q \) are also greater than 0. Without loss of generality, we assume that \( \lambda \) is the minimal one.
From (5), one gets
\[ \dot{V} \leq -\lambda (e_{2x}^2 + e_{3x}^2 + e_{4x}^2) \leq 0. \]  
(6)

Based on the Lyapunov stability theory, we obtain \( \lim_{t \to \infty} V = 0 \), which implies that \( \lim_{t \to \infty} e_{2x} = \lim_{t \to \infty} e_{3x} = \lim_{t \to \infty} e_{4x} = 0 \). The Proof of Theorem 1 is completed.

IV. THE CONTROL SCHEME OF THE NEW CHAOTIC SYSTEM

The control goal considered in this section is to force the trajectories of system (1) to converge to the origin. For this end, we add a controller \( u \) to system (1) and get
\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= a_1 x_1 + b_1 x_2 + c_1 x_3 + d_1 x_4 + l_1 x_1^2 + l_2 x_3^3, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= a_2 x_1 + b_2 x_2 + c_2 x_3 + d_2 x_4 + u,
\end{align*}
\]  
(7)

where \( u \) is the controller.

Note that the available state variable of system (7) is \( x_1 \). In order to design an appropriate controller \( u \) we need an observer to recover the unavailable states \( x_2, x_3, x_4 \).

Based on system (2), the observer of system (7) is:
\[
\begin{align*}
\hat{x}_2 &= w + (k + b_1) x_1, \\
\dot{w} &= -k \hat{x}_2 + c_1 \hat{x}_3 + d_1 \hat{x}_4 + a_1 x_1 + l_1 x_1^2 + l_2 x_3^3, \\
\dot{\hat{x}}_3 &= \hat{x}_4, \\
\dot{\hat{x}}_4 &= a_2 x_1 + b_2 \hat{x}_2 + c_2 \hat{x}_3 + d_2 \hat{x}_4 + u,
\end{align*}
\]  
(8)

where \( \hat{x}_2, \hat{x}_3, \) and \( \hat{x}_4 \) are the estimated values of \( x_2, x_3, \) and \( x_4 \), respectively, and \( k > 0 \) is a constant to be designed later.

Prior to giving our further result, we introduce some lemmas which will be used in the proofs of the following Theorems.

**Lemma 1** For system (7), if \( \lim_{t \to \infty} (c_1 x_3 + d_1 x_4) = 0 \), then \( \lim_{t \to \infty} x_3 = \lim_{t \to \infty} x_4 = 0 \).

The proof of Lemma 1 is similar to that of Lemma 2.1.1 in [15], which is omitted here.

**Lemma 2** [16] If \( \lim_{t \to +\infty} \alpha(t) = 0 \), then the origin of the system
\[ \dot{x} = -\lambda x + \alpha(t) \]

is globally asymptotically stable, where \( \lambda > 0 \).

From Lemma 1, it is easy to see that in order to show \( \lim_{t \to +\infty} x_i = 0, i = 1, 2, 3, 4 \) we only need to prove \( \lim_{t \to +\infty} (x_1 + x_2) = 0, \lim_{t \to +\infty} (c_1 x_3 + d_1 x_4) = 0 \). For this end we take the following Lyapunov candidate:
\[ V_1(t) = \frac{(x_1 + x_2)^2}{2} + \frac{\alpha_1^2}{2}, \]  
(9)
where $\alpha_1 = (a_1 + 1)x_1 + (b_1 + 2)x_2 + c_1x_3 + d_1x_4 + l_1x_1^2 + l_2x_1^3$.

The derivative of (9) is

$$
\dot{V}(t) = (x_1 + x_2)(x_2' + \hat{x}_2) + \alpha_1((a_1 + 1)x_1 + (b_1 + 2)x_2'
+c_1\hat{x}_3 + d_1\hat{x}_4 + 2l_1x_1\hat{x}_1 + 3l_2x_1^2\hat{x}_1
= -(x_1 + x_2)^2 + (x_1 + x_2)\alpha_1 + \alpha_1((a_1 + 1)x_1 + (b_1 + 2)x_2'
+c_1\hat{x}_3 + d_1\hat{x}_4 + 2l_1x_1\hat{x}_1 + 3l_2x_1^2\hat{x}_1
= -(x_1 + x_2)^2 - \alpha_1^2 + \alpha_1(a_1 + x_1 + x_2 + (a_1 + 1)x_1 + (b_1 + 2)x_2'
+c_1\hat{x}_3 + d_1\hat{x}_4 + 2l_1x_1\hat{x}_1 + 3l_2x_1^2\hat{x}_1
= -(x_1 + x_2)^2 - \alpha_1^2 + \alpha_1(a_1 + x_1 + x_2 + (a_1 + 1 + 2l_1x_1 + 3l_2x_1^2)x_2
+(b_1 + 2)(a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 + l_1x_1^2 + l_2x_1^3) + c_1x_4
+d_1(a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4 + d_1x_1a_2 + d_1x_2b_2)
$$

where $\beta_1 = a_1 + x_1 + x_2 + (a_1 + 1 + 2l_1x_1 + 3l_2x_1^2)x_2 + (b_1 + 2)(a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 + l_1x_1^2 + l_2x_1^3) + c_1x_4 + d_1c_2x_4 + d_1d_2x_4 + d_1x_1a_2 + d_1x_2b_2$.

If we take

$$
u = -\frac{1}{d_1} \hat{\beta}_1,
$$

where $\hat{\beta}_1 = (a_1 + 1)x_1 + (b_1 + 2)x_2 + c_1\hat{x}_3 + d_1\hat{x}_4 + l_1x_1^2 + l_2x_1^3 + x_1 + \hat{x}_2 + (a_1 + 1 + 2l_1x_1 + 3l_2x_1^2)\hat{x}_2
+(b_1 + 2)(a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 + l_1x_1^2 + l_2x_1^3) + c_1\hat{x}_4 + d_1c_2\hat{x}_3 + d_1d_2\hat{x}_4 + d_1x_1a_2 + d_1x_2b_2$,

then Equation (10) becomes

$$
\dot{V}(t) \leq -(x_1 + x_2)^2 - \alpha_1^2 + \alpha_1(\beta_1 - \hat{\beta}_1) = -2V(t) + \alpha_1(\beta_1 - \hat{\beta}_1).
$$

It is noted that $\alpha_1$ is bounded and $\lim_{t \to \infty} (\beta_1 - \hat{\beta}_1) = 0$, thus $\lim_{t \to \infty} \alpha_1(\beta_1 - \hat{\beta}_1) = 0$. According to Lemma 2, it is obvious that $\lim_{t \to \infty} (x_1 + x_2) = 0$, $\lim_{t \to \infty} \alpha_1 = 0$, which implies that $\lim_{t \to \infty} (x_1 + x_2) = 0$, $\lim_{t \to \infty} (c_1x_3 + d_1x_4) = 0$. In view of Lemma 1, we have $\lim_{t \to \infty} x_i = 0, i = 1, 2, 3, 4$.

Based on the above discussion, we have the following Theorem.

**Theorem 2.** If the control law $u$ is designed by (11) and the unavailable states $x_2, x_3, x_4$ are estimated by system (8), then the origin of system (7) is globally asymptotically stable which implies that $\lim_{t \to \infty} x_i = 0, i = 1, 2, 3, 4$. 
V. THE SYNCHRONIZATION SCHEME OF THE NEW CHAOTIC SYSTEM

In our drive-response configure, we suppose system (1) is the drive system, then the controlled response system is given as

\[
\begin{align*}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= a_1 y_1 + b_1 y_2 + c_1 y_3 + d_1 y_4 + l_1 y_1^2 + l_2 y_1^3, \\
\dot{y}_3 &= y_4, \\
\dot{y}_4 &= a_2 y_1 + b_2 y_2 + c_2 y_3 + d_2 y_4 + u,
\end{align*}
\]

where \( u \) is the controller to be designed later.

In order to obtain further results, we make the following assumption.

**Assumption 3.** Suppose the available state variables of system (12) is \( y_1 \).

Assumption 3 means that the controller \( u \) in system (12) is a function of \( x_1, y_1, \hat{x}_i, \hat{y}_i \), where \( \hat{x}_i, \hat{y}_i \) are the estimated values of \( x_i, y_i, i = 2, 3, 4 \), respectively. The states \( \hat{x}_i, i = 2, 3, 4 \) are given by observer (2) and the states \( \hat{y}_i, i = 2, 3, 4 \) are obtained by the following observer:

\[
\begin{align*}
\dot{\hat{x}}_2 &= w + (k + b_1) y_1, \\
\dot{\hat{x}}_3 &= -k \hat{x}_2 + c_1 \hat{x}_3 + d_1 \hat{y}_4 + a_1 y_1 + l_1 y_1^2 + l_2 y_1^3, \\
\dot{\hat{y}}_4 &= a_2 \hat{y}_1 + b_2 \hat{y}_2 + c_2 \hat{y}_3 + d_2 \hat{y}_4 + u,
\end{align*}
\]

where \( \hat{x}_2, \hat{x}_3, \) and \( \hat{y}_4 \) are the estimated values of \( y_2, y_3, \) and \( y_4 \), respectively, and \( k > 0 \) is designed as in Equation (3).

By defining the synchronization error \( e_i = y_i - \hat{x}_i, i = 1, 2, 3, 4 \), the synchronization error system is achieved as follows:

\[
\begin{align*}
\dot{e}_1 &= e_2, \\
\dot{e}_2 &= a_1 e_1 + b_1 e_2 + c_1 e_3 + d_1 e_4 + l_1 y_1^2 + l_2 y_1^3 - (l_1 \hat{x}_1^2 + l_2 \hat{x}_1^3), \\
\dot{e}_3 &= e_4, \\
\dot{e}_4 &= a_2 e_1 + b_2 e_2 + c_2 e_3 + d_2 e_4 + u,
\end{align*}
\]

It is clear that the synchronization problem between systems (1) and (12) is replaced by the equivalent problem of stabilizing system (14).

For the purpose of designing a controller \( u \) such that \( \lim_{\tau \to +\infty} e_i = 0, i = 1, 2, 3, 4 \), we consider the following Lyapunov function:

\[
V_1(t) = \frac{(e_1 + e_2)^2}{2} + \frac{a_2^2}{2},
\]

where \( a_2 = (a_1 + 1) e_1 + (b_1 + 2) e_2 + c_1 e_3 + d_1 e_4 + l_1 (y_1^2 - \hat{x}_1^2) + l_2 (y_1^3 - \hat{x}_1^3) \).
The derivative of (15) is

$$\dot{V}_1(t) = (e_1 + e_2)(e_2 + \hat{e}_2) + \alpha_2((a_1 + 1)\hat{e}_1 + (b_1 + 2)\hat{e}_2 + c_1\hat{e}_3 + d_1\hat{e}_4 + 2l_1(y_1\dot{y}_1 - x_1\dot{x}_1) + 3l_2(y_1^2\dot{y}_1 - x_1^2\dot{x}_1))$$

$$= -(e_1 + e_2)^2 + (e_1 + e_2)\alpha_2 + \alpha_2((a_1 + 1)\hat{e}_1 + (b_1 + 2)\hat{e}_2 + c_1\hat{e}_3 + d_1\hat{e}_4 + 2l_1(y_1\dot{y}_1 - x_1\dot{x}_1) + 3l_2(y_1^2\dot{y}_1 - x_1^2\dot{x}_1))$$

$$= -(e_1 + e_2)^2 + \alpha_2(e_1 + (a_1 + 2)e_2 + (b_1 + 2)\hat{e}_2 + c_1\hat{e}_3 + d_1\hat{e}_4$$

$$+ 2l_1(y_1\dot{y}_1 - x_1\dot{x}_1) + 3l_2(y_1^2\dot{y}_1 - x_1^2\dot{x}_1))$$

$$= -(e_1 + e_2)^2 - \alpha_2^2 + \alpha_2(e_1 + (a_1 + 2)e_2 + (b_1 + 2)\hat{e}_2 + c_1\hat{e}_3 + d_1\hat{e}_4$$

$$+ d_1(a_2e_1 + b_2e_2 + c_2e_3 + d_2e_4 + u)$$

$$+ 2l_1(y_1\dot{y}_2 - x_1\dot{x}_2) + 3l_2(y_1^2\dot{y}_2 - x_1^2\dot{x}_2))$$

$$= -(e_1 + e_2)^2 - \alpha_2^2 + \alpha_2(\hat{\beta}_2 + d_1u),$$

where $\hat{\beta}_2 = \alpha_2 + e_1 + (a_1 + 2)e_2 + (b_1 + 2)\hat{e}_2 + c_1\hat{e}_3 + d_1\hat{e}_4 + d_1(a_2e_1 + b_2e_2 + c_2e_3 + d_2e_4 + 2l_1(y_1y_2 - x_1x_2) + 3l_2(y_1^2y_2 - x_1^2x_2).$

If we take

$$u = -\frac{1}{d_1}\hat{\beta}_2,$$

where $\hat{\beta}_2 = (a_1 + 1)e_1 + (b_1 + 2)\hat{e}_2 + (a_1 + 2)e_2 + c_1\hat{e}_3 + d_1\hat{e}_4 + l_1(y_1^2 - x_1^2) + l_2(y_1^2 - x_1^2) + c_1\hat{e}_4 + d_1(a_2e_1 + b_2e_2 + c_2e_3 + d_2e_4 + 2l_1(y_1\dot{y}_2 - x_1\dot{x}_2) + 3l_2(y_1^2\dot{y}_2 - x_1^2\dot{x}_2))$ and $\hat{e}_i = y_i - \dot{x}_i, \ i = 2, 3, 4,$ then Equation (16) becomes

$$\dot{V}_1(t) \leq -(e_1 + e_2)^2 - \alpha_2^2 + \alpha_2(\hat{\beta}_2 - \hat{\beta}_2) = -2V_1(t) + \alpha_2(\hat{\beta}_2 - \hat{\beta}_2).$$

It is noted that $\alpha_2$ is bounded and $\lim_{t \to +\infty} \beta(\hat{\beta}_2 - \hat{\beta}_2) = 0,$ thus $\lim_{t \to +\infty} \alpha_2(\hat{\beta}_2 - \hat{\beta}_2) = 0.$

According to Lemma 3, it is obvious that $\lim_{t \to +\infty} (e_1 + e_2) = 0,$ $\lim_{t \to +\infty} \alpha_2 = 0,$ which implies that $\lim_{t \to +\infty} (e_1 + e_2) = 0,$ $\lim_{t \to +\infty} (c_1e_3 + d_1e_4) = 0.$ In view of Lemma 1, we have $\lim_{t \to +\infty} e_i = 0, \ i = 1, 2, 3, 4.$

Based on the above discussion, we have the following Theorem.

**Theorem 3.** Under Assumptions 1–3, if the control law $u$ is designed by (17) and the observer for systems (1) and (12) are respectively given by systems (2) and (13), then the origin of system (14) is globally asymptotically stable, which implies that $\lim_{t \to +\infty} (y_i - x_i) = 0, \ i = 1, 2, 3, 4,$ i.e., the synchronization between systems (1) and (12) is achieved.

**VI. SIMULATION AND RESULTS**

In this section, numerical simulation results are presented to demonstrate the effectiveness of the proposed control and synchronization methods. The parameter values
considered in the numerical simulations correspond to chaotic behavior [14] and these are:

\[ a_1 = 2, \quad b_1 = -0.22, \quad c_1 = 0.11, \quad d_1 = 0.05, \quad a_2 = -10.28, \quad b_2 = 0.05, \quad c_2 = -0.11, \quad d_2 = -0.05. \]

The parameters \( k \) and \( l \) are chosen as \( k = 100 \) and \( l = 0.01 \) such that Equation (3) holds.

**Example 1.** The first example is the control of the new chaotic system. In numerical simulations, the initial values of the system (7) and system (8) are taken as

\[ (x_1(0), x_2(0), x_3(0), x_4(0)) = (1, 2, 3, 1) \]

and

\[ (\hat{x}_2(0), \hat{x}_3(0), \hat{x}_4(0)) = (-4, 5, -6), \]

respectively. Thus, the initial errors are

\[ (\hat{e}_2(0), \hat{e}_3(0), \hat{e}_4(0)) = (-6, 2, -7). \]

The simulation results are shown in Figs. 2–3. The time response of the states \( x_1, x_2, x_3, x_4 \) of system (7) is shown in Fig. 2. Fig. 3 displays the time evolution of the errors \( \hat{e}_2, \hat{e}_3, \hat{e}_4 \) between the system (7) and observer (8).

**Example 2.** The second example is the synchronization between two identical new chaotic systems. In numerical simulations, the initial values of the systems (1), (2), (12), and system (13) are taken as

\[ (x_1(0), x_2(0), x_3(0), x_4(0)) = (1, 2, 3, 1), \]

\[ (\hat{x}_2(0), \hat{x}_3(0), \hat{x}_4(0)) = (4, -5, 6), \]

\[ (y_1(0), y_2(0), y_3(0), y_4(0)) = (9, 8, 10, 12), \]

and

\[ (\hat{y}_2(0), \hat{y}_3(0), \hat{y}_4(0)) = (3, -4, 2), \]

respectively. Thus, the initial errors are

\[ (\hat{e}_2(0), \hat{e}_3(0), \hat{e}_4(0)) = (2, -8, 5), \]

and

\[ (\hat{e}_2(0), \hat{e}_3(0), \hat{e}_4(0)) = (-5, -14, -10). \]

The simulation results are shown in Figs. 4–6. Fig. 4 is the picture of the time evolution of the synchronization errors between systems (1) and (12). Fig. 5 displays the time evolution of the errors \( \hat{e}_2, \hat{e}_3, \hat{e}_4 \) between the system (1) and observer (2). The time response of the errors \( \hat{e}_2, \hat{e}_3, \hat{e}_4 \) between the system (12) and observer (13) is shown in Fig. 6.
FIG. 3: The time evolution of the errors $\hat{e}_{2x}$, $\hat{e}_{3x}$, $\hat{e}_{4x}$ between the system (7) and observer (8).

FIG. 4: The synchronization errors between systems (1) and (12).
FIG. 5: The time evolution of the errors $\hat{e}_{2x}, \hat{e}_{3x}, \hat{e}_{4x}$ between the system (1) and observer (2).

FIG. 6: The time response of the states $\hat{e}_{2y}, \hat{e}_{3y}, \hat{e}_{4y}$ between the system (12) and observer (13).
VII. CONCLUSION

This paper investigates the control and synchronization of the new chaotic system via a single input. We assume that only one state variable, i.e., $x_1$ or $y_1$ is available for the new chaotic system. Based on constructing a proper observer, some novel control conditions are derived via the Lyapunov stability theory. In our control scheme we don’t need to use the chaotic system’s unavailable states, thus our approach is more realistic and practical than most of the existing ones, since they assume that all the states are available. Finally, numerical simulations are carried out to verify the efficiency of the proposed method.

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