Analytical Evolution of Displaced Thermal States for Amplitude Damping

Wei-Feng Wu$^{1,2,}$

$^1$Department of Mechanical and Electronic Engineering, Chizhou University, Chizhou, Anhui, 247000, China
$^2$Interdisciplinary Research Center of Quantum Information and Photoelectric Information, Chizhou, Anhui, 247000, China

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A displaced thermal state (DTS) was first introduced by Jeong et al. [Phys. Rev. Lett. 97, 100401 (2006)]; its superpositions can be used in the transfer of nonclassicality. In this paper, the analytical evolution law of the DTS exposed to dissipative environments is obtained according to the Kraus operator for amplitude damping. The results manifestly show that, after undergoing amplitude damping, the initial DTSs still remain mixed and thermal with the exponential decay due to the amplitude damping. An appealing feature is that the entropy evolution in this process completely depends on the mixedness.

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I. INTRODUCTION

In quantum optics, coherent states and thermal states have played important roles in the aspects of both theoretical and experimental physics. As an intermediate state between the thermal state and the coherent state, a displaced thermal state (DTS) was introduced by Jeong and Ralph [1]; its superpositions are used to realize the transfer of nonclassicality. From Ref. [1] the integral representation of a DTS is given by

$$\rho_{DTS}(M, d) = \int d^2 \beta P(M, d; \beta) |\beta\rangle \langle \beta|,$$  \hspace{1cm} (1)

where $|\beta\rangle = \exp(-|\beta|^2/2 + \beta a^\dagger)|0\rangle$ is the coherent state with amplitude $\beta$ [2, 3], and $P(M, d; \beta) = \frac{2}{\pi(M-1)} e^{-2|\beta-d|^2/(M-1)}$ represents a Gaussian noise with mixedness $M$ and displacement $d$ in the phase space. The mixedness $M$ increases with an increase of the temperature $T$ of the thermal field, i.e., $e^{\hbar \omega/T} = (M + 1)/(M - 1)$, where $\hbar$ is Planck's constant and $\omega$ refers to the frequency of the thermal field. Through using the technique of integration within an ordered product (IWOP) of operators [4] and the operator identity $|0\rangle \langle 0| = e^{-a^\dagger a}$, [5], as well as the mathematical integration formula [6]

$$\int \frac{d^2 \alpha}{\pi} \exp(-g |\alpha|^2 + h \alpha + s \alpha^*) = \frac{1}{g} \exp \left( \frac{hs}{g} \right)$$ \hspace{1cm} (2)

$^*$Electronic address: wuweifeng78@sina.com

to perform the integration (1), we have

\[
\rho_{DTS}(M,d) = \frac{2}{M+1} \cdot \exp \left[ -\frac{2}{M+1}(|d|^2 - d\alpha^† - d^* \alpha + \alpha^† \alpha) \right],
\]

(3)

where the symbol \( : : \) notes the normal ordering of the operators (that is, all creation operators are to the left of all annihilation operators and the creation and annihilation operators are commutative within the normal ordering symbol). Next, we shall consider as several special cases the DTS with only special given values of \( M \) and \( d \). When \( M = 1 \), \( \rho_{DTS}(1,d) = |d\rangle \langle d| \) is a pure coherent state, and \( \rho_{DTS}(M,0) = \frac{2}{3M+1} e^{a^† a \ln \frac{M-1}{M+1}} \) corresponds to a thermal field having mixedness \( M \) in the case of \( d = 0 \). However, for the case of both \( M = 1 \) and \( d = 0 \), \( \rho_{DTS}(M,d) \) becomes vacuum.

This paper shall investigate the analytical evolution rule of the DTS for amplitude damping: the photon number decay and evolution law of the density operator and entropy, based on the infinite operator-sum representation (Kraus operator) of the density operator \( \sum(t) \), since it can give the explicit evolution law of any initial state in the amplitude damping channel. Especially, to the best of my knowledge, the entropy evolution of the TSs for amplitude damping has not been previously reported, the reason may lie in that the analytical evolution of density operators of the DTSs in this process cannot be derived. The remainder of this paper is arranged as follows: In Section II the time-evolution of the density operator of a DTS for amplitude damping is obtained via the continuous-variable entangled state representation (CESR). Sections III and IV respectively present the photon number decay and Wigner function’s evolution based on the density operator’s evolution of a DTS in this channel. Finally, the evolution of entropy for amplitude damping is also demonstrated.

**II. EVOLUTION OF A DTS FOR AMPLITUDE DAMPING**

For any open system, noise always accompanies the fundamental dynamics processes when the systems interact with the environment. Among various real environments, amplitude damping is an important cause for the non-classicality deterioration of the systems. In the interaction picture, the evolution of the density operator can be described by the standard master equation of the form [7]

\[
\frac{d}{dt} \rho(t) = \kappa (2\alpha a^† - a^† \alpha - \alpha \rho),
\]

(4)

which refers to the transfer of energy from the system to a zero temperature environment, \( \kappa \) is the cavity decay rate. To obtain the evolution law of the DTSs for amplitude damping, we first recall CESR in the enlarged Fock space, i.e., \( |\eta\rangle = D(\eta)|\eta = 0\rangle \) [5], where \( |\eta = 0\rangle = \exp(\alpha^† \tilde{a}^†) |0,0\rangle \) and \( D(\eta) = \exp(\eta a^† - \eta^* a) \) is the displacement operator, \( \tilde{a}^† \) is a fictitious mode corresponding to the real photon creation operator \( a^† \), \( \tilde{a} \) can annihilate \( |0\rangle \). It should be emphasized that the state \( |\eta\rangle \) is just the common eigenvector of \( (a - \tilde{a}^†) \) and \( (a^† - \tilde{a}) \).
Using the well-behaved properties of $|\eta = 0\rangle$ and the scalar product $\langle \eta | \rho \rangle$, as well as the over-complete relation of $|\eta\rangle$, we can obtain the infinite operator-sum representation of $\rho(t)$ \[^8\], i.e.,

$$
\rho(t) = \sum_{n=0}^{\infty} \mathcal{M}_n \rho(0) \mathcal{M}_n^\dagger,
$$

(5)

where $\rho(0)$ is the density operator for the initial state, and $\mathcal{M}_n = \sqrt{T_n} e^{-\kappa t a a^\dagger}$ refers to the Kraus operator corresponding to $\rho(t)$. $T = 1 - e^{-2\kappa t}$. For convenience of calculation, we first investigate how the density operator $\rho_{DTS}(M,d)$ evolves with time $t$ under the above damping channel in terms of the Kraus operator $\mathcal{M}_n$. From Eq. (5) we see that the bosonic annihilation operator $a^n$ stands on the left of $\rho(0)$ and meanwhile the creation operator $a^\dagger n$ just resides on its right side, so introducing the anti-normal ordering form of $\rho_{DTS}(M,d)$ can facilitate the derivation of the evolution law of $\rho_{DTS}(M,d)$. Using Eqs. (2), (3) and the scalar product $\langle -\beta | \beta \rangle = \exp(-2|\beta|^2)$, as well as the integration formula for calculating the anti-normal ordering product, we have

$$
\rho_{DTS}(M,d) = \frac{2}{M-1} \exp \left[ \frac{2}{M-1} (-|d|^2 + da^\dagger + d^* a - a^\dagger a) \right] ;
$$

(6)

which is just the anti-normally ordered product of $\rho_{DTS}(M,d)$ ($; ;$ refers to anti-normal ordering). Thus

$$
\rho_{DTS}(M,d; t) = \frac{2}{M-1} \sum_{n=0}^{\infty} \frac{T^n}{n!} e^{-\kappa t a a^\dagger} \exp \left[ -\frac{2}{M-1} (|d|^2 - da^\dagger - d^* a + a^\dagger a) \right] ; a^\dagger n e^{-\kappa t a a^\dagger}.
$$

(7)

Substituting the completeness relation for coherent states into (7) and using the operator identity $e^{-\kappa t a a^\dagger} |z\rangle = e^{-|z|^2/2} e^{za^\dagger e^{-\kappa t}} |0\rangle$, we have

$$
\rho_{DTS}(M,d; t) = \frac{2}{M-1} \sum_{n=0}^{\infty} \frac{T^n}{n!} \int \frac{d^2 z}{\pi} |z|^{2n} \exp \left[ -\frac{2}{M-1} (|d|^2 - dz^* - d^* z + |z|^2) \right. \\
- |z|^2 + za^\dagger e^{-\kappa t} + z^* a e^{-\kappa t} - a^\dagger a \left. \right] .
$$

(8)

Further, using the mathematical integral formula \[^6\]

$$
\int \frac{d^2 \alpha}{\pi} \alpha^m \alpha^* n \exp(h|\alpha|^2 + sa + \eta \alpha^*) = e^{-sn/h} \sum_{l=0}^{\min(m,n)} \frac{m!n! s^{n-l} \eta^{m-l}}{l!(m-l)!(n-l)!} (-h)^{m+n-l+1}
$$

(9)
and the Laguerre polynomial \( L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (-x)^{n-k} \) as well as the relation \( \sum_{n=0}^{\infty} z^n L_n(y) = \frac{1}{1-e^{-yz/(1-z)}} \), Eq. (8) becomes

\[
\rho_{\text{DTS}}(M, d; t) = \frac{2}{2 + (M - 1)e^{-2\kappa t}} \exp \left[ - \frac{2}{2 + (M - 1)e^{-2\kappa t}} (|d|^2 e^{-2\kappa t} - d^* e^{-\kappa t} a - de^{-\kappa t} a^\dagger + a^\dagger a) \right] : ,
\]

which means that the DTS still remains mixed and thermal but the parameters \( d \) and \( M \) are closely related to the decay factor \( e^{-\kappa t} \) after undergoing amplitude damping. In particular, in the case of \( M = 1 \), Eq. (10) corresponds to the evolution of a coherent state \( \rho_{\text{DTS}}(1, d) \). At the initial time \( \kappa t = 0 \), \( \rho_{\text{DTS}}(M, d; t) \) agrees with the result of the normally-ordered product of a DTS in Eq. (7). In the limit \( \kappa t \to \infty \), \( \rho_{\text{DTS}}(M, d; t) \) decays to vacuum.

Further, using the relation \( e^{\lambda a^\dagger a} = : \exp [(e^\lambda - 1)a^\dagger a] : \), we find

\[
\rho_{\text{DTS}}(M, d; t) = f e^{-f|d|^2 e^{-2\kappa t}} \exp \left[ fde^{-\kappa t} a^\dagger \right] \exp \left[ a^\dagger a \ln(1 - f) \right] \exp \left[ f d^* e^{-\kappa t} a \right]
\]

with \( f = \frac{2}{2 + (M - 1)e^{-2\kappa t}} \). It is clear from Eq. (11) how a DTS \( \rho_{\text{DTS}}(M, d) \) evolves into a superposition of photon-added thermal state \( \rho_{\text{DTS}}(M, d; t) \) due to the existence of amplitude damping.

### III. EVOLUTION OF THE NUMBER OF PHOTONS

Now we shall calculate the evolution of the number of photons for an initial state \( \rho_{\text{DTS}}(M, d) \) in an amplitude damping channel:

\[
n_{\text{DTS}}(M, d; t) = Tr[\rho_{\text{DTS}}(M, d; t) a^\dagger a] = f e^{-f|d|^2 e^{-2\kappa t}} Tr \left[ \exp \left( fde^{-\kappa t} a^\dagger \right) \exp \left[ a^\dagger a \ln(1 - f) \right] \exp \left( f d^* e^{-\kappa t} a \right) a^\dagger a \right].
\]

Further, using the operator identities

\[
e^{\eta a^\dagger e^{-\eta a}} = (a^\dagger + \eta)^\dagger
\]

and

\[
e^{a^\dagger a \ln \xi} e^{-a^\dagger a \ln \xi} = \xi a^\dagger a
\]

(13) and

(14)
we can express the evolution formula for the number of photons as

\[ n_{DTS}(M, d; t) = f e^{-f|d|^2} e^{-2\kappa t} Tr \left[ \exp \left( f d e^{-\kappa t} a^\dagger \right) (1 - f) a^\dagger + f d^* e^{-\kappa t} \right] \exp \left[ a^\dagger a \ln(1 - f) \right] \exp \left( f d^* e^{-\kappa t} a \right) \right] \]

\[ = f e^{-f|d|^2} e^{-2\kappa t} \int \frac{d^2 z}{\pi} \langle z | [(1 - f) a^\dagger + f d^* e^{-\kappa t}] a \exp \left( f d e^{-\kappa t} a^\dagger - f a^\dagger a + f d^* e^{-\kappa t} a \right) : |z \rangle \]

\[ = f e^{-f|d|^2} e^{-2\kappa t} \int \frac{d^2 z}{\pi} (1 - f) |z|^2 + f d^* e^{-\kappa t} z \exp \left( -f |z|^2 + f d^* e^{-\kappa t} z + f d e^{-\kappa t} z^* \right) \]

\[ = M - 1 + |d|^2 \] \( e^{-2\kappa t} \), \quad (15) \]

which indicates that the photon number decays exponentially. On one hand, for \( d = 0 \), \( n_{DTS}(M, 0; t) = n e^{-2\kappa t} \) with the average photon number \( n = \frac{M - 1}{2} \) of a thermal field with the mixedness \( M \). On the other hand, when \( M = 1 \), \( n_{DTS}(1, d; t) = |d|^2 e^{-2\kappa t} \) corresponding to the evolution of the number of photons for amplitude damping in an initial coherent state.

### IV. WIGNER FUNCTION EVOLUTION

In quantum optics, the Wigner quasi-probability distribution is very useful since its partial negativity usually provides a good indication of the non-classicality of quantum states. Using the coherent state representation of the Wigner operator \( \Delta(z, z^*) = \int \frac{d^2 z'}{\pi} |z + z'| \langle z' - z' | e^{2z'^* - z'^*} \rangle [9] \), where \( |z \rangle \) is the coherent state and the scalar product of two coherent states is \( \langle z'| z \rangle = \exp[-\frac{1}{2}(|z|^2 + |z'|^2) + z^* z] \), the Wigner function for the DTS can be calculated as

\[ W_{DTS}(M, d; z, t) = Tr[\rho_{DTS}(M, d; t) \Delta(z, z^*)] \]

\[ = f \int \frac{d^2 z'}{\pi^2} \langle z - z' | : \exp \left( -f [\frac{|d|^2}{2} e^{-2\kappa t} - d^* e^{-\kappa t} a - a d^* e^{-\kappa t} a^\dagger + a^\dagger a] \right) : | z + z' \rangle e^{z'^* - z'^*} \]

\[ = f \int \frac{d^2 z'}{\pi^2} \exp(-f |d|^2 e^{-2\kappa t} + f d^* e^{-\kappa t} z + f d e^{-\kappa t} z^*) \exp(-f z^* - z'^* - 2 |z'|^2) \]

\[ = \frac{f}{\pi(2 - f)} \exp \left( -\frac{2f}{2 - f} \left( |z|^2 + |d|^2 e^{-2\kappa t} - d^* e^{-\kappa t} z - d e^{-\kappa t} z^* \right) \right) \]

\[ = \frac{1}{\pi(1 + (M - 1)e^{-2\kappa t})} \exp \left( -\frac{2}{1 + (M - 1)e^{-2\kappa t}} (|z|^2 + |d|^2 e^{-2\kappa t} - d^* e^{-\kappa t} z - d e^{-\kappa t} z^*) \right) \]. \quad (16) \]

In particular, in the case of \( d = 0 \),

\[ W_{DTS}(M, 0; z, t) = \frac{1}{\pi(1 + (M - 1)e^{-2\kappa t})} \exp \left( -\frac{2}{1 + (M - 1)e^{-2\kappa t}} |z|^2 \right) \], \quad (17) \]
which is the Wigner function evolution for the thermal field in the amplitude damping channel. However, when \( M = 1 \),

\[
W_{DTS}(1, d; z, t) = \frac{1}{\pi} \exp \left[ -2(|z|^2 + |d|^2 e^{-2\kappa t} - d^* e^{-\kappa t} z - de^{-\kappa t} z^*) \right],
\]

which refers to the analytical evolution of the coherent state for amplitude damping. On the other hand, the limit behavior of \( W_{DTS}(M, d; z, t) \) for \( t \to \infty \) is of interest since \( W_{DTS}(M, d; z, t \to \infty) \sim \frac{1}{\pi} e^{-2|z|^2} \) corresponds to the vacuum.

V. ENTROPY EVOLUTION IN THE AMPLITUDE DAMPING CHANNEL

In the quantum information framework, the von Neumann entropy as a quantum extension of Shannon entropy has been extensively used in entanglement measures. Given the density operator \( \sum \), the von Neumann entropy may be defined as \( S(\rho) = -Tr[\rho \ln \rho] \).

It is clear from the above equation that the known natural exponential expression of \( \rho \) is convenient for calculating the von Neumann entropy \( S(\rho) \). Using the operator theorem \( \exp A \exp B = \exp (A + \mu B + \nu) \) which holds for \([A, B] = \mu B + \nu\), the three natural exponentials in (11) may be written as a single exponential as follows:

\[
\rho_{DTS}(M, d; t) = f \exp[|d|^2 e^{-2\kappa t} \ln(1 - f)] \exp \left\{ [a^\dagger a - (da^\dagger + d^* a) e^{-\kappa t}] \ln(1 - f) \right\}.
\]

Thus the natural logarithm of \( \rho_{DTS}(M, d; t) \) can be calculated as

\[
\ln \rho_{DTS}(M, d; t) = \ln f + |d|^2 e^{-2\kappa t} \ln(1 - f) + [a^\dagger a - (da^\dagger + d^* a) e^{-\kappa t}] \ln(1 - f).
\]

It then follows that the von Neumann entropy of \( \rho_{DTS}(M, d; t) \) is given by

\[
S(\rho_{DTS}(M, d; t)) = -Tr[\rho_{DTS}(M, d; t) \ln \rho_{DTS}(M, d; t)]
\]

\[
= -\ln f - |d|^2 e^{-2\kappa t} \ln(1 - f) - f e^{-f|d|^2 e^{-2\kappa t}} \ln(1 - f)
\]

\[
\times Tr \left\{ e^{fde^{-\kappa t}a^\dagger a \ln(1 - f)} e^{fde^{-\kappa t}a^\dagger a} [a^\dagger a - (da^\dagger + d^* a) e^{-\kappa t}] \right\}.
\]
Owing to the relation
\[
\begin{align*}
\text{Tr} \left\{ e^{fde^{-\kappa t}a'} e^{a^\dagger a\ln(1-f)} e^{d^* e^{-\kappa t}a^\dagger a - (da^\dagger + d^* a)e^{-\kappa t}} \right\} \\
= \text{Tr} \left\{ e^{fde^{-\kappa t}a'} [(1 - f)a^\dagger + f d^* e^{-\kappa t}a_1 \ln(1-f)] e^{d^* e^{-\kappa t}a^\dagger a - (da^\dagger + d^* a)e^{-\kappa t}} \right\} \\
= \text{Tr} \left\{ e^{fde^{-\kappa t}a'} e^{-\kappa t}d^* e^{-\kappa t}a_1 \ln(1-f) e^{d^* e^{-\kappa t}a^\dagger a} \right\}
\end{align*}
\]

\[
\begin{align*}
= \int \frac{d^2 \alpha'}{\pi} \langle \alpha' \rangle e^{fde^{-\kappa t}a'} e^{-\kappa t}d^* e^{-\kappa t}a_1 \ln(1-f) e^{d^* e^{-\kappa t}a^\dagger a - (da^\dagger + d^* a)e^{-\kappa t}} - f |d|^2 e^{-2\kappa t} \rangle : |\alpha'\rangle \\
= \int \frac{d^2 \alpha'}{\pi} \left[(1 - f)[|\alpha'|^2 - (da^\dagger + d^* \alpha')e^{-\kappa t}] - f |d|^2 e^{-2\kappa t} \right] \\
\exp(-f |\alpha'|^2 + f d^* e^{-\kappa t} \alpha' + fde^{-\kappa t} a^\dagger a) \\
= \frac{1}{f}[1 - f - |d|^2 e^{-2\kappa t}] e^{f|d|^2 e^{-2\kappa t}},
\end{align*}
\]

(23)

thus we have

\[
S(\rho_{DTS}(M, d; t)) = -\ln \frac{2}{2 + (M - 1)e^{-2\kappa t}} - \frac{M - 1}{2} e^{-2\kappa t} \ln \frac{(M - 1)e^{-2\kappa t}}{2 + (M - 1)e^{-2\kappa t}},
\]

(24)

which indicates that entropy evolution for the DTS for amplitude damping is directly related to the mixedness \(M\) with the exponential decay but cannot completely rely on the displacement \(d\) in the phase space; this is a remarkable result. As a special case, \(\kappa t = 0\), we find

\[
S(\rho_{DTS}(M, d; 0)) = -\ln \frac{2}{M + 1} - \frac{M - 1}{2} \ln \frac{M - 1}{M + 1},
\]

(25)

which corresponds to the entropy of a thermal field with mixedness \(M\). In the limit of large \(\kappa t\), \(S(\rho_{DTS}(M, d; t \to \infty)) \to 0\).

In conclusion, based on the Kraus operator corresponding to amplitude damping, this paper has obtained the evolution law of a DTS as an intermediate state between thermal and coherent in this channel. It clearly finds that the initial DTSs still remain mixed and thermal with the exponential decay after interacting with amplitude damping. More interestingly, photon number decay and Wigner function evolution in this channel are directly related to the mixedness \(M\) of a Gaussian noise and the displacement \(d\) in the phase space, however the entropy evolution only depends on the noise mixedness \(M\).

References


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