Atomic Evolution in a Driven Gravitational Cavity

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We propagate an atomic wavepacket in a gravitational cavity which is modulated by means of an external periodic driving force. We show that in classical evolution it displays classical chaos, whereas in the quantum domain the wavepacket exhibits dynamical localization. Moreover, we also find dynamical revivals in the quantum evolution and prove that this is a generic characteristics of all driven systems.

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I. Introduction

The presence of periodically vibrating surfaces in a system poses interesting questions about its characteristics. In general such systems possess more than one degree of freedom and, therefore, exhibit chaotic evolution in both the classical and quantum domains [1, 2]. In this paper we consider optical surfaces which undergo periodic vibrations as a function of time and study the dynamics of atoms on them.

Total internal reflection of laser light produces an evanescent wave field on the dielectric surface. Atoms moving towards this field experience an exponentially increasing force and are reflected back [3]. We provide a periodic modulation to the evanescent wave field by means of an acousto-optic modulator [4]. Atoms, bouncing under gravity on the modulated evanescent wave field, have a bounded motion. Therefore such a system is called a modulated gravitational cavity or Fermi accelerator [5] for the bouncing atoms. This system is of particular interest in the field of atom optics [6].

We show that, in the long time evolution which consists of many bounces of the atoms, they display dynamical localization. This localization of atoms is a prototype of the Anderson localization, as observed in solid state physics [7]. Moreover, the atoms exhibit the presence of dynamical revivals [8, 9]. We prove that this property is generic for all periodically modulated systems.

Our paper is arranged as follows: In section II we explain the model of the experiment. In section III we discuss the characteristics of dynamical localization, whereas we discuss the presence of dynamical revivals in section IV and give an analytical expression for the revival time. In section V, we summarize the results.
II. Fermi accelerator: driven gravitational cavity

Consider a cloud of atoms initially trapped and cold in a magneto-optical trap (MOT) down to the micro-Kelvin scale. The MOT is placed at a certain height above the evanescent wave mirror. The mirror for the atoms results from the total internal reflection of the laser light field from the surface of the glass prism. The atomic mirror is modulated by providing an intensity modulation to the evanescent light field via an acousto-optic modulator. The polarization of the evanescent wave field is inside the plane of the reflection. We consider the atomic dynamics only along the $\hat{z}$-axis, that is, along the normal to the surface of the mirror.

We consider the limit of large detuning, leading to a very little probability of finding the atom in the excited state. Therefore, we neglect the possibility of spontaneous emission and take an atom to be in the ground state. Far above the surface of the prism the evanescent light field is negligible and the atom undergoes an attractive constant gravitational force, $F_{gr} = \dot{z} = mgz = M gz$. Here, $M$ is the atomic mass and $g$ denotes the gravitational acceleration. In the region of the evanescent wave field, the atom experiences an exponentially changing force, $F_{opt} = v_0 k e^{kz}$. The force due to the optical field is repulsive on the atom, which is in the ground state. It is due to blue detuning of the laser light field from the transition frequency of the atom. Hence, the gravitational field together with the exponential field makes a cavity for the atom, as shown in Fig. 2. The atomic wavepacket exhibits bounded motion inside this cavity and displays quantum carpets in its evolution [10], as a function of time.
FIG. 2. The potential seen by an atom in a gravitational cavity (left) and its evolution over half of the revival time (right): On switching off the MOT, the atom starts its motion from an initial height at $t = 0$. It moves under the influence of the gravitational field towards the evanescent wave atomic mirror. The atom experiences a constant attractive force towards the mirror, $F_{gr} = mg$, due to the linear gravitational field $V_{gr} = mgz$, where $m$ is the mass of the atom and $g$ is the gravitational acceleration. Close to the surface of the mirror the effect of the evanescent light field is dominant. The atom experiences an exponential force $F_{opt} = V_0 \cdot e^{-z}$ away from the mirror, due to the exponentially decaying potential $V_{opt} = V_0 e^{-z}$. Both the potentials together make the gravitational cavity for the atom. On the left side we display the evolution of the wavepacket in position space as a function of time. The wavepacket displays quantum carpets in its evolution.

In order to study the effect of an external driving force on this elementary quantum system, the evanescent wave is assumed to be modulated by an acusto-optical modulator [4], that is $E_0 = E_0 e^{i \sin \omega t}$. We consider that the optical driving frequency $\omega$ is well detuned from any atomic resonance and also take into account the symmetry of the problem in the $(x; y)$-plane. Hence, the effective one-dimensional center-of-mass motion of an atom moving towards the atomic mirror due to the gravitational potential is governed by the Hamiltonian [11]

$$H = \frac{p^2}{2M} + M gz + \frac{\alpha f}{4} e^{-2Lz - \epsilon + \epsilon \sin \omega t}.$$  

Here, $p$ is the atomic center-of-mass momentum. The effective Rabi frequency $\frac{\alpha f}{4}$ characterizes the strength of the influence of the applied electric field.

III. Dynamical localization

Throughout this paper we study the dynamics of an atom of mass $M$ moving with momentum $p$ in a time dependent potential $V(z; t)$. Figure 3 compares the classical distributions in position space and in momentum space with their quantum counterparts. We recognize a dramatic difference between these distributions. The classical position and momentum distributions represented by thick lines follow the exponential barometric formula

$$P_{cl}(z) = \frac{1}{k_B T} \exp \left( \frac{\mu}{k_B T} \right)$$
FIG. 3. Dynamical localization in the atomic Fermi accelerator. Comparison between the quantum mechanical (thin lines) and classical (thick lines) momentum (a) and position (b) distributions, on a logarithmic scale. The quantum momentum distribution exhibits an exponential localization, in sharp contrast to the corresponding classical Gaussian distribution. Whereas the classical position distribution changes linearly, the corresponding quantum distribution is narrower, and follows a square root law. The thin dashed lines indicate the corresponding fits. The linear dependence of the classical distribution results from the linear potential in the Boltzmann distribution. For more details and the values of the parameters see Ref. [13]. Here, $x = \frac{w^2}{g}z$ and $\tilde{p} = \frac{w^2}{m g}p$.

and the Gaussian distribution

$$P_{cl}(p) = p \frac{1}{2^{3/2}k_B T} \exp \left( -\frac{\mu}{2M k_B T} \right) i; \tag{3}$$

shown in Fig. 3(a) by thick dashed lines. Here $T$ is temperature, which increases with time due to the modulation of the mirror.

In contrast, the quantum mechanical momentum distribution shown in Fig. 3(a) by a thin solid line is exponential rather than Gaussian. By analogy with the kicked rotor model, we estimate the momentum wave function to be exponentially localized

$$\tilde{A}(p) \propto e^{ip\varepsilon}; \tag{4}$$

where $\varepsilon$ describes the localization length. We display this exponential fit in Fig. 3(a) by the thin dashed line.

However, in position space we estimate the position wave function by

$$\tilde{A}(z) \propto e^{ipz\varepsilon}; \tag{5}$$

and display this fit in Fig. 3(b) by a thin dashed line.

We conclude this subsection by emphasizing that dynamical localization in a modulated gravitational cavity occurs only in a window of modulation [11]. This feature makes the Fermi accelerator even more interesting in the study of dynamical localization.
IV. Quantum recurrences

In the moving coordinates we may write the time dependent Schrödinger equation for an atom in the modulated gravitational cavity as [12]

\[ i\hbar\dot{\hat{A}} = \frac{1}{2M} \frac{\partial^2}{\partial z^2} + Mz(g_i + !^2 \sin \omega t) + V_0 e^{ikz} \hat{A}, \]  

(6)

where, \( z = z - \frac{c}{2} \sin \omega t \). Here, we are considering \( V_0 = -\varepsilon f = \mathcal{A} \), the steepness \( \varepsilon \) \( 2! = \mathcal{C} \) and the modulation strength \( \varepsilon ^2 = \left( \frac{2!}{2! \mathcal{L}} \right) \).

A convenient way of obtaining insight into the influence of the acousto-optical external driving force on the atomic dynamics is to investigate the evolution of an atomic center-of-mass wave packet. For this purpose, we consider a Gaussian wave packet,

\[ \hat{A}(0) = \left( 2\pi \mathcal{C} z^2 \right)^{1/4} \exp \left( -\frac{\mu (z - z_0)^2}{(2\mathcal{C} z)^2} \right) \exp \left( -i\frac{\mu p_0 z}{\mathcal{A}} \right) \]  

(7)

at \( t = 0 \) and propagate it in the gravitational cavity for different modulation strengths, \( \varepsilon \). Here, \( z_0 \) describes the average position, \( p_0 \) denotes the average momentum of the wave packet and \( \mathcal{C} z \) is the spatial uncertainty. In Fig. 4 characteristic time dependences of the autocorrelation function, \( C(t) \) \( \hat{A}\langle 0 \rangle \hat{A}\langle t \rangle \), of the wave packet are shown for \( z_0 = 20:1 \ \text{m} \), \( \mathcal{C} z = 0:28 \ \text{m} \) and \( p_0 = 0 \). Without any external periodic perturbation (uppermost figure), the well known scenario of revivals and fractional revivals [14, 15] is clearly apparent. In the presence of a sufficiently weak external periodic driving force the revivals and fractional revivals are still observable. However, the revivals decrease in magnitude and the revival time exhibits a pronounced dependence on the external driving force.

The potential, generated by the gravitational acceleration and by the evanescent laser field, has the approximate form of a time dependent triangular potential well, like in the Fermi accelerator [5]. This, in the subsequent, approximate treatment we replace this potential by the idealized, simpler form of a triangular well with an infinitely high potential barrier at the position \( z = \frac{c^2}{2! \mathcal{L}} \sin \omega t \) of the mirror.

We provide an analytical treatment to the problem by calculating the time of revival of the initial wavepacket in the driven system with the help of secular perturbation theory. We find that the time of revival in the presence of an external modulation, \( T_\varepsilon \), is connected to the revival time in the absence of modulation, \( T_0 \), by

\[ T_\varepsilon = T_0 \exp \left( \frac{1}{2} \frac{8\mathcal{C} E_N}{\mu} \frac{\mu I_0}{I_N} \frac{3\mathcal{C}^2 + 1}{(2\mathcal{C}^2 + 1)^3} \right) \]  

(8)

Here, \( I_0 \) and \( I_N \) are the classical actions at the phase point corresponding to the initial condition and the phase point corresponding to the center of \( N \)th resonance, respectively. On substituting the value of \( \mathcal{C} \) [12], calculated at \( l = l_0 \), in Eq. (8) we get

\[ T_\varepsilon = T_0 \exp \left( \frac{1}{8} \frac{M g}{E_{N_0}} \frac{3(1_i r)^2 + \mathcal{A}^2}{((1_i r)^2 \mathcal{A}^2)^3} \right) \]  

(9)
FIG. 4. The revival phenomena in a gravitational cavity in the absence and in the presence of periodic modulation: The autocorrelation of a Gaussian wave packet of a cesium atom as a function of time, prepared at \( t = 0 \) with \( \xi Z = 0.28 \ \text{m} \), for \( \xi = 0 \) (a), \( \xi = 14.3 \ \text{nm} \) (b), \( \xi = 28.7 \ \text{nm} \) (c), and \( \xi = 57.4 \ \text{nm} \) (d). The parameters are \( ! = 2\sqrt{\xi} \ 0.93 \ \text{KHz} \), \( \text{eff} = 23.38 \ \text{KHz} \), and \( \cdot = 0.57 \ \text{m} \). The average position of the wave packet in the gravitational cavity is \( z_0 = 20.1 \ \text{m} \), which corresponds to the mean quantum number \( n_0 = 176.16 \). Here we use scaled time \( E' = \omega t \).

where, \( r'(E_N - E_{n_0})^{1/2} \) and \( a'(r^2)! = 4E_{n_0} \). If the initial energy is large, \( i.e., E_{n_0} \gg ! \), we may consider \( a^2 \) much smaller than \( (1 + r)^2 \), which leads to

\[
T_0 = T_0 \ 1 \ i \ \frac{3}{8} \ \frac{1}{E_{n_0}} \ \frac{1}{(1 + r)^4} \quad (10)
\]

Our analytical result explains in a simple way the quantitative dependence of the revival time \( T_0 \) on the characteristic parameters of the problem, namely the driving frequency \( ! \) and the driving amplitude \( \cdot \). In order to access the accuracy of these results we calculate the revival time \( T_0 \) by integrating the Schrödinger equation, Eq. (6), numerically, and compare it with the analytically obtained result of Eq. (10). For this comparison we have considered two different initial conditions of the atomic wave packet above the surface of the atomic mirror. In Fig. 5 the solid line with circles corresponds to (a) \( z_0 = 29.8 \ \text{m} \), which implies a state with mean principle quantum number \( n_0 = 322.51 \), however, the solid line with squares corresponds to (b) \( z_0 = 20.1 \ \text{m} \), which implies \( n_0 = 176.16 \). In both cases the initial average momentum is \( p_0 = 0 \). The first initial condition lies further away from the center of the corresponding primary resonance as compared to the second one.
FIG. 5. We compare the ratio between the revival time for a driven system and the revival time of an undriven system, $T_r/T_0$, for two different initial conditions, $z_0 = 29.8 \, \text{m}$ (solid line with circles) and $z_0 = 20.1 \, \text{m}$ (solid line with squares). The comparison comes from analytical results based on Eq. (9) (solid lines) and from exact numerical calculations of the time dependent Schrödinger equation, Eq. (6) (dashed lines). All the other parameters are the same as those in Fig. 4. Here we define $\bar{\lambda} = \frac{w^2}{g}$. 

Numerically we find that the change in revival time depends quadratically on the strength of the external modulation $\bar{\lambda}$, as predicted from Eq. (10). We plot the numerically obtained result for the two initial conditions in Fig. 5 with the help of dashed lines. We find that the change of the revival time is smaller in the case (a) than in the case (b). Our analytical result explains this dependence: In case (a) our chosen initial condition, $z_0 = 29.8 \, \text{m}$, has a higher energy $E_{n_0}$ than in case (b), where $z_0 = 20.1 \, \text{m}$. Since the change in revival time has an inverse dependence on the square of the energy $E_{n_0}$, as a result, we observe a smaller change in the revival time when the average energy is larger, as in case (a), and a larger change in revival time when the average energy is smaller, as in (b), as shown in Fig. 5.

V. Summary

We have demonstrated that the dynamics of a material wave packet in a periodically driven gravitational cavity exhibits dynamical localization and quantum mechanical revivals. We have presented the first results on the quantitative dependence of the revival time on the characteristic parameters of the problem, namely the driving frequency and the driving strength. We show that these dependences can be understood quantitatively in a satisfactory way by using semiclassical
secular theory.

In view of the recent experimental [16] developments the presented quantitative predictions are accessible to experimental observation. With the help of these characteristics we may explain the generic behavior of driven systems [17].

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