Quantum Theory of Some Two-dimensional Oscillator Problems in Complex Coordinates

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We present a unified approach for calculating the transformation matrix of a class of two-dimensional oscillation problems in complex coordinates as a quantum theory. These problems are characterized by the fact that their classical equations of motion are linear differential equations with constant coefficients in the complex coordinates only. They can be solved in a way no different from that of one-dimensional linear oscillation problems.

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I. INTRODUCTION

Oscillation problems in one dimension [1] are an interesting subject in classical mechanics as well as in other regimes of science. Of these, linear oscillation theory is the easiest to describe in terms of mathematics. The equation of motion is a linear differential equation with constant coefficients which is soluble and is well studied and understood. In two dimensions there are interesting problems where the equations of motion are given as linear differential equations with constant coefficients in the complex variable \( w = x_1 + ix_2 \) only. That is, we can solve these two-dimensional problems using an analogous method from one-dimensional linear oscillation theory, and the complex conjugate solution comes out naturally by conjugation. So we would say that the problem is just one complex dimensional. As expected, the quantum theory of these two-dimensional problems should be easy to deal with in complex coordinates. We shall discuss in this paper a unifying method to study these two-dimensional oscillation problems quantum mechanically in complex coordinates.

In this article we try to study the quantum theory of these oscillation problems in a most succinct way. We shall calculate the transformation function [2–4], which encodes the information about the energy spectrum and all the energy eigen-functions into one characteristic function. Feynman [5] had reinterpreted this function as the path integral. Here we employ a method of finding the infinitesimal variation of this function and try to integrate this variation to get the transformation function. This method is easy to carry out here because the classical equations of motion are soluble in terms of the exponential functions, and different cases can be studied with the same techniques.

This paper is organized as follows. In Section II we begin a study of the simplest case, the free particle in two dimensions, implying that the frequency \( \omega \) takes the limit to zero. This is done to set up the notation and the recipe. In Section III the standard isotropic harmonic oscillator in two dimensions is studied. In Section IV we study the
Landau problem, the electron in a constant magnetic field $B$ in the symmetric gauge. In this gauge, rotational invariance is maintained. We show that the same procedure as before will yield easily the expression for the transformation function. In Section V we extend the problem to include a harmonic trap as well as a transverse electric field. In Section VI we just draw our conclusion with a relevant discussion.

II. FREE PARTICLE

It seems appropriate here to begin with the study of the simplest oscillator problem, i.e., the free particle problem. We present some details here in order to straighten out the notation, which has some peculiarities for the complex coordinates. The Lagrangian for this case can be written as

$$L = \frac{m}{2} \left( \dot{x}_1^2 + \dot{x}_2^2 \right) = \frac{m}{2} \dot{w} \dot{\bar{w}},$$

(1)

where we denote $w = x + iy$ to be the coordinates of the position vector. We look upon the complex coordinates $w$ and $\bar{w}$ as a coordinate transformation of $x_1$ and $x_2$. The generalized momentum is $\bar{p} = \frac{\partial L}{\partial \dot{w}} = \frac{1}{2} m \dot{\bar{w}}$, that is $\bar{p} = \frac{1}{2} (p_1 - ip_2)$. A similar calculation will give the complex conjugate $p$. The Hamiltonian function will then be

$$H = \bar{p} \dot{w} + p \dot{\bar{w}} - L = \frac{2}{m} \bar{p} p.$$  

(2)

For the quantum theory we have to take the position and momentum as operators so that the commutator $[w, \bar{p}]$ is nonzero, and actually $[w, \bar{p}] = i\hbar$ and $[w, p] = 0$. We do not care about the ordering in the Hamiltonian here because $\bar{p}$ and $p$ commute. There are various ways to formulate the quantum theory. For the present problem we adopt the formulation of Schwinger [3] for a detailed study. We have to calculate the transformation function $\langle w''; t''|w'; 0 \rangle$, which encodes the full details of the quantum theory. This function denotes the amplitude of transition of the state $w = w'$ at $t = 0$ to the state $w = w''$ at $t = t''$. Indeed, by standard manipulations we can show that the transformation function is just equivalent to the Feynman path integral [5]. However, the calculation of this transformation is done here a l’a Schwinger directly via the method of solving the differential equation for its total variation. We have the time evolution equation

$$|w, t\rangle = e^{\frac{i}{\hbar} H t} |w, 0\rangle.$$  

(3)

In this coordinate representation the momentum operator generates translation as

$$\frac{\partial}{\partial w} \langle w, t\rangle = \frac{i}{\hbar} \bar{p} \langle w, t\rangle.$$  

(4)

Take a variation of the transformation function and we get a general formula

$$\delta \langle w'', t''|w', 0 \rangle = \frac{i}{\hbar} \langle w'', t''| \bar{p}'' \delta w'' + p'' \delta \bar{w}'' - \bar{p}' \delta w' - p' \delta \bar{w}' - H \delta t''|w', 0 \rangle.$$  

(5)
To proceed for this problem we have to make use of the equations of motion

\[
\dot{w} = \frac{2}{m} p, \\
\dot{p} = 0,
\]

which yield

\[
w(t) = w' + \frac{2}{m} p't, \\
p(t) = p',
\]

that is

\[
w'' = w' + \frac{2}{m} p'' t, \\
p'' = \frac{m (w'' - w')}{{2t''}}.
\]

Taking into mind of the noncommutativity of the operator equation

\[
[w', \bar{w}''] = \frac{2i\hbar}{m} t'',
\]

we can calculate the variation as

\[
\langle w'', t'' | w', 0 \rangle = \frac{2i\hbar}{m} t'' \exp \left[ \frac{i\hbar}{2} \frac{m (\bar{w}' - \bar{w}') (w'' - w')}{{2t''}} \right] - \frac{\delta t''}{t''},
\]

where the last term on the R.H.S. comes from the noncommutativity equation. Integrating, we get

\[
\langle w'', t'' | w', 0 \rangle = \frac{C}{i^m} \exp \left[ \frac{i}{\hbar} \frac{m (\bar{w}' - \bar{w}') (w'' - w')}{{2t''}} \right].
\]

The integration constant \( C \) can be determined to be \( \frac{m}{2\pi i\hbar} \) by the delta function normalization. By standard procedures we can retrieve the energy spectrum and all the energy eigen-functions from this transformation function. The free particle transformation function here also serves as the normalization condition for the other oscillation problems when we take the limit of zero frequency.

### III. ISOTROPIC HARMONIC OSCILLATOR

We turn next to the standard oscillation problem, the isotropic harmonic oscillator. In standard notation the Hamiltonian can be written as

\[
H = \frac{2}{m} \bar{p} p + \frac{1}{2} m \omega^2 \bar{w} w.
\]
The equations of motion are
\[
\dot{w} = \frac{2}{m} p, \\
\dot{p} = -\frac{1}{2} m \omega^2 w.
\]
(16)
(17)

These are coupled first order linear differential equations in \( w \) and \( p \) with constant coefficients in the phase space. It is easy to decouple the differential equations by taking an appropriate linear combination of \( w \) and \( p \). This manipulation yields the solutions
\[
p(t) + \frac{m \omega}{2} i w(t) = \left( p' + \frac{m \omega}{2} i w' \right) e^{i \omega t},
\]
\[
p(t) - \frac{m \omega}{2} i w(t) = \left( p' - \frac{m \omega}{2} i w' \right) e^{-i \omega t},
\]
that is,
\[
w'' = w' \cos \omega t'' + \frac{2p'}{m \omega} \sin \omega t'',
\]
\[
p'' \sin \omega t'' = \frac{m \omega}{2} \left( w'' \cos \omega t'' - w' \right).
\]
(18)
(19)
(20)
(21)

It is good to check that if we replace \( t'' \) by \( -t'' \) we just interchange the initial and final conditions. The commutator relation is as follows:
\[
[w', \bar{w}'] = \frac{2i \hbar}{m \omega} \sin \omega t''.
\]
(22)

By using Eq. (5) we can compute the transformation function as
\[
\frac{\delta(w'', t''|w', 0)}{\langle w'', t''|w', 0 \rangle} = i \frac{m \omega}{\hbar} \left[ \frac{2 \sin \omega t''}{2 \sin \omega t''} \left( (\bar{w}' w'' + \bar{w} w') \cos \omega t'' - (\bar{w}'' w' + w'' \bar{w}') \right) \right] \frac{\cos \omega t''}{\sin \omega t''} \delta(\omega t'').
\]
(23)

Integrating the variation, and taking care of the proper normalization, we can write the transformation function as
\[
\langle w'', t''|w', 0 \rangle = \frac{m \omega}{2 \pi i \hbar \sin \omega t''} \exp \left[ i \frac{m \omega}{\hbar} \left( (\bar{w}' w'' + \bar{w} w') \cos \omega t'' - (\bar{w}'' w' + w'' \bar{w}') \right) \right].
\]
(24)

IV. LANDAU PROBLEM IN THE SYMMETRIC GAUGE

Another problem in the same category is the problem of an electron in a transverse constant magnetic, i.e., the Landau problem [6] in the symmetric gauge. We begin with the Lagrangian
\[
L = \frac{1}{2} m \ddot{w} - \frac{eB}{4i} (\dot{w} \dot{w} - \ddot{w} \ddot{w}),
\]
(25)
where $e$ is the magnitude of the charge for the electron, and $B$ is the constant magnetic field out of the plane. This represents an electron minimally coupled to the magnetic interaction. We represent $eB/m = \omega_c$ as the cyclotron frequency. The generalized momentum is $\vec{p} = \partial L/\partial \dot{\omega} = \frac{1}{2} m \dot{\omega} + \frac{eB}{4} i \dot{\omega}$, and the Hamiltonian is

$$H = \frac{1}{m} \left[ \left( \vec{p} - \frac{eB}{4} i \dot{\omega} \right) \left( p + \frac{eB}{4} i \omega \right) + \left( p + \frac{eB}{4} i \omega \right) \left( \vec{p} - \frac{eB}{4} i \dot{\omega} \right) \right].$$

We would like to make a remark on the form of the Hamiltonian. The round bracket terms are just proportional to $\dot{\omega}$ and $\dot{\omega}$, but they do not commute. So to make the Hamiltonian to correspond to the sum of total kinetic energies we have to take a symmetric sum of the product of the two terms.

The equations of motion are

$$\dot{\omega} = \frac{2}{m} \left( p + \frac{eB}{4} i \omega \right), \quad \dot{p} = \frac{eB}{2m} i \left( p + \frac{eB}{4} i \omega \right),$$

yielding

$$p(t) + \frac{eB}{4} i \omega(t) = \left( p' + \frac{eB}{4} i \omega' \right) e^{i \omega_c t},$$

$$p(t) - \frac{eB}{4} i \omega(t) = p' - \frac{eB}{4} i \omega'. \quad (29)$$

This describes an electron moving with the cyclotron frequency $\omega_c$ in a circle around the guiding center, which can be anywhere depending on the initial conditions. Solving for $\omega''$ and $p''$, we get

$$\frac{eB}{2} i \omega'' = \frac{eB}{4} i \omega' \left( e^{i \omega_c t'} + 1 \right) + p' \left( e^{i \omega_c t''} - 1 \right),$$

$$p'' = \frac{ieB}{4(1 - e^{-i \omega_c t''})} \left( \omega'' \left( 1 + e^{-i \omega_c t''} \right) - 2 \omega' \right).$$

Therefore we can compute the commutator as

$$[\omega', \omega''] = \frac{2 \hbar}{eB} (1 - e^{-i \omega_c t''}). \quad (33)$$

It is now straightforward to calculate the variation of the transformation function, and the result is

$$\frac{\delta \langle \omega'', t'' | \omega', 0 \rangle}{\langle \omega'', t'' | \omega', 0 \rangle} = \frac{i eB}{\hbar} \delta \left[ \frac{(\omega'' \omega'' + \omega' \omega')}{\tan \frac{\omega'' t''}{2}} - \frac{e^{i \omega_c t''} \omega'' \omega' + e^{-i \omega_c t''} \omega' \omega''}{\sin \frac{\omega'' t''}{2}} \right] - \cos \frac{\omega'' t''}{2} \delta (\omega_c t''). \quad (34)$$
This variation can be integrated to give
\[
\langle w'', t'' | w', 0 \rangle = \frac{m \omega_c}{4 \pi i \hbar \sin \frac{\omega''}{2}} \exp \left[ \frac{i m \omega_c}{\hbar} \left( \frac{\langle \tilde{w}'' - \tilde{w}' \rangle (w'' - w') - i (\tilde{w}' w'' - \tilde{w}'' w')} {2 \tan \frac{\omega''}{2}} \right) \right].
\]

(34)

V. OTHER RELATED PROBLEMS

We would expect that the same procedure can be applied to calculate the transformation function for the Landau problem in a harmonic trap characterized by the Hamiltonian
\[
H = \frac{1}{m} \left[ \left( \tilde{p} - \frac{eB}{4} i \tilde{w} \right) \left( p + \frac{eB}{4} i w \right) + \left( p + \frac{eB}{4} i w \right) \left( \tilde{p} - \frac{eB}{4} i \tilde{w} \right) \right] + \frac{1}{2} m \omega^2 \tilde{w} w.
\]

(35)

This is just a hybrid of the isotropic harmonic oscillator problem and the Landau problem. Skipping the details of computation, we just write down the expression for the transformation function as
\[
\langle w'', t'' | w', 0 \rangle = \frac{m \Omega}{2 \pi i \hbar \sin \Omega t''} \exp \left[ \frac{i m \Omega}{\hbar} \left( \frac{\tilde{w}'' w'' + \tilde{w}' w'} {\tan \Omega t''} - \frac{e^i \omega'' \tilde{w}'' w' + e^{-i} \omega'' \tilde{w}' w''} {\sin \Omega t''} \right) \right],
\]

(36)

where we have denoted \( \Omega = \sqrt{\frac{\omega^2}{4} + \omega^2} \). A discussion of the same problem in the formalism of raising and lowering operators can be found in [7].

Classically [8], adding a transverse electric field to the problem of an electron in a constant magnetic field is soluble. There is then a drift of the guiding center transverse to the electric field as the particular solution. So we expect that we can also solve the Landau problem with an electric field \( \mathcal{E} = E_x + i E_y \) on the transverse plane. The Hamiltonian can be chosen to be
\[
H = \frac{1}{m} \left[ \left( \tilde{p} - \frac{eB}{4} i \tilde{w} \right) \left( p + \frac{eB}{4} i w \right) + \left( p + \frac{eB}{4} i w \right) \left( \tilde{p} - \frac{eB}{4} i \tilde{w} \right) \right] - \frac{1}{2} (\tilde{E} w + \mathcal{E} \tilde{w}).
\]

(37)

The only subtlety is that we now have, in addition, a particular solution for the inhomogeneous term of the linear equations of motion. This is analogous to the one-dimensional case where the problem of a particle under the effect of a constant force is soluble. Everything should work out following the same procedure. As a final remark the Landau problem with a harmonic trap and a transverse electric field should also be soluble along the same line.
VI. DISCUSSION AND CONCLUSION

We have presented a unified way of calculating the transformation matrix for the quantum theory of a class of two-dimensional linear oscillation problems. The special feature of these two-dimensional problems is that their equations of motion in the phase space can be written as first-order linear differential equations with constant coefficients in the complex variables \( w \) and \( p \) only. So we can adapt the method used in the one-dimensional oscillator problem without any modification and the calculations are straightforward and methodical. As a consequence, the final forms for the transformation matrix for the different problems are very similar in form, just with some minor variations. All these calculations are done in a methodical and pedagogical way.

Now that this article demonstrates that using the complex coordinates in these two-dimensional oscillation problems yields an easy computation for the transformation matrix. It seems reasonable enough to extend using the complex coordinates for the same problems when working in the other formulations, like solving for the Schrödinger equations. It is likely to yield soluble special functions in terms of the complex coordinates, without separating the Schrödinger equations into Cartesian coordinates or polar coordinates. Also it would be nice to consider the symmetry of these quantum problems using the complex coordinates, as we believe that symmetry begets simplicity and the inherent symmetry is linked with our choice of coordinates. This symmetry, as hinted from the Landau problem, may be the blending of the translational symmetry and the rotational symmetry as well as the gauge symmetry, at least.

References