Theory and Numerics of Double-Vibrational Resonance in the Overdamped Oscillator

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Vibrational resonance (VR) in the overdamped oscillator with a triple-well potential driven by a low-frequency force and a high-frequency force is investigated by using the method of direct separation of motion. An approximate analytical expression for the response amplitude \( Q \) is obtained. Based on the approximate analytical expression, the double VR is found. The positions of \( B \) (the amplitude of the high-frequency signal) and \( \Omega \) (the frequency of the high-frequency signal) at which the VR occur are determined, and the relation between them are analyzed. The theoretical predictions are found to be in good agreement with the numerical results.

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I. INTRODUCTION

In recent years, a great deal of interest has been shown in research on vibrational resonance (VR) in nonlinear systems. VR is when a resonant behavior appears in some nonlinear systems driven by a low-frequency force \( A \cos(\omega t) \) and a high-frequency force \( B \cos(\Omega t) \), and the optimal amplitude of high-frequency driving enhances the response of an excitable system to a low-frequency subthreshold signal. Originally it was found by Landa and McClintock [1]. Stochastic resonance (SR) is a general phenomenon in nonlinear dynamical systems, and is widely studied. There are three indispensable factors in the above stochastic systems: two or more stable states, an input signal (which may contain a weak periodic signal) and noise [2]. VR closely resembles SR, in that a high-frequency harmonic signal plays the role of noise. However, there is an essential difference between SR and VR. In the first case, noise changes both the effective stiffness and the damping factor of the system, whereas in the latter case the high-frequency vibration changes only the effective stiffness [3].

Two-frequency signals are widely applied in many fields. For example, the occurrence of resonant behavior with respect to a low-frequency force caused by a high-frequency force and an analytical treatment for it in a bistable system are shown in Ref. [1, 4]. A single and double resonances in a double-well Duffing oscillator [5], signal transmission by VR in one-way coupled bistable systems [6], and delay induced quasi-periodic VR [7, 8] have been reported. SR control based on VR has been presented [9]. Vibrational and stochas-

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tic resonances in two coupled overdamped anharmonic oscillators driven by an amplitude modulated force was also studied [10]. Moreover, experimental evidence in an analog simulation of the overdamped Duffing oscillator [11] and in an optical system [12] have also been presented. VR in a noise-induced structure has been studied in Refs. [13, 14]. Moreover, it is important to mention that two frequency signals are also an object of intensive interest in other fields, including laser physics [15], acoustics [16], neuroscience [17], and the physics of the ionosphere [18]. Specially in the biological field, VR in neuron populations has been investigated [19, 20].

Recently, various nonlinear phenomena have been studied in the quintic oscillator. For example the effect of time delay in the overdamped quintic oscillator in the presence of additive and multiplicative noises has been analyzed by Jia [21]. The occurrence of chaos has been studied in the parametrically driven triple-well system in Ref. [22]. SR in an overdamped nonlinear system with a triple-well potential [23] and splitting of the Kramers escape rate were analyzed [24, 25]. Specially, VR in a damped quintic oscillator with monostable potentials has been studied [26, 27]. However, VR in the overdamped oscillator with a triple-well potential was not analyzed, which is the main motivation of this work.

In this work, VR in an overdamped oscillator with a triple-well potential driven by a low-frequency force and a high-frequency force is investigated by using the method of direct separation of the motion and numerical simulation. In Sec. II, VR in the overdamped oscillator with a triple-well potential is investigated from the theoretical point of view. In Sec. III, the validity of the approximate method is checked by numerical simulation. Brief conclusions are given in Sec. IV.

II. THEORETICAL DESCRIPTION OF VIBRATIONAL RESONANCE

The motion equation of the overdamped oscillator with a triple-well potential and two periodic forces is given by

$$\gamma \dot{x} = -U'(x) + A \cos(\omega t) + B \cos(\Omega t).$$  \hspace{1cm} (1)

Where the term $B \cos(\Omega t)$ is a high-frequency force with amplitude $B$ and frequency $\Omega$. $A \cos(\omega t)$ is a low-frequency signal with amplitude of $A$ and frequency $\omega$. Assuming that $\Omega \gg \omega$ and $A \ll B$. $\gamma$ is the dissipation constant. $U(x)$ is given by [25]

$$U(x) = x^2 \left( bx^2 - c \right)^2,$$  \hspace{1cm} (2)

which is a symmetric triple-well potential, $b$ and $c$ are the parameters of the potential, which are chosen so that $b > 0$ and $c > 0$, or $b < 0$ and $c < 0$. The symmetric triple-well potential is shown by Fig. 1.

By using the method of direct separation of the motion described in Refs. [5, 28], for $\Omega \gg \omega$ the solution of Eq. (1) consists of a slow motion $X(t)$ with period $2\pi/\omega$ and a fast motion $\Psi(t, \Omega t)$ with period $2\pi/\Omega$ (or period $2\pi$ in the fast time). The mean value of the
fast motion is $\Psi_{av} = \frac{1}{2\pi} \int_0^{2\pi} \Psi \, d\tau = 0$. Substituting $x = X + \Psi$ into Eq. (1), we obtain the following set of equations of motion:

$$
\gamma \dot{X} + 6 b^2 X^5 + (60 b^2 \Psi_{av}^2 - 8 b c) X^3 + 60 b^2 X^2 \Psi_{av}^3 \\
+ (-24 b c \Psi_{av}^2 + 2 c^2 + 30 b^2 \Psi_{av}^4) X - 8 b c \Psi_{av}^3 \\
+ 6 b^2 \Psi_{av}^5 = A \cos(\omega t),
$$

(3)

where $\Psi_{av} = (1/2\pi) \int_0^{2\pi} \Psi \, d\tau$, $i = 1, 2, \ldots, 5$. Remembering that $\Psi$ is a rapidly changing force, we assume that $\dot{\Psi} \gg \Psi, \Psi^2, \Psi^3, \Psi^4, \Psi^5$ and neglect all the terms in the left-hand side of Eq. (4) except the term $\gamma \dot{\Psi}$. We obtain $\Psi = \frac{B}{\Omega} \sin(\omega t)$, $\Psi_{av} = 0$, $\Psi_{av}^2 = \frac{B^2}{2\gamma\Omega^2}$, $\Psi_{av}^3 = 0$, $\Psi_{av}^4 = \frac{3B^4}{8\gamma^2\Omega^4}$, and $\Psi_{av}^5 = 0$. Eq. (3) becomes

$$
\gamma \dot{X} + C_1 X + C_2 X^3 + C_3 X^5 = A \cos(\omega t),
$$

(5)

where

$$
C_1 = -\frac{12 b c B^2}{\gamma^2 \Omega^2} + 2 c^2 + \frac{45 b^2 B^4}{4 \gamma^4 \Omega^4}, \quad C_2 = \frac{30 b^2 B^2}{\gamma^2 \Omega^2} - 8 b c, \quad C_3 = 6 b^2.
$$

(6)
The effective potential corresponding to the slow motion of the system described by Eq. (5) is

\[ V_{\text{eff}} = \frac{1}{2} C_1 X^2 + \frac{1}{4} C_2 X^4 + \frac{1}{6} C_3 X^6. \]  

(7)

The effective potential \( V_{\text{eff}} \) depends on the parameters \( b, c, B, \Omega, \) and \( \gamma \). Consequently, by varying \( b, c, B, \Omega, \) and \( \gamma \), new equilibrium states can be created, or the number of equilibrium states can be reduced.

The equilibrium points about which slow oscillations take place can be calculated from Eq. (7) and are as follows:

\[ X_1^* = 0, \quad X_{2,3}^* = \pm \left[ \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_3} \right]^{1/2}, \]

\[ X_{4,5}^* = \pm \left[ \frac{-C_2 - \sqrt{C_2^2 - 4C_1C_3}}{2C_3} \right]^{1/2}. \]  

(8)

Suppose \( C_1 > 0 \) and \( C_2 < 0 \) with \( C_2^2 > 4C_1C_3 \), the effective potential is a triple-well potential.

Next, we obtain the equation for the deviation of the slow motion \( X \) from an equilibrium point \( X^* \) by substituting \( Y = X - X^* \) into Eq. (5):

\[ \gamma \dot{Y} + \beta_1 Y + \beta_2 Y^2 + \beta_3 Y^3 + \beta_4 Y^4 + \beta_5 Y^5 = A \cos(\omega t), \]

(9)

where

\[ \beta_1 = C_1 + 3C_2 X^* + 5C_3 X^4, \quad \beta_2 = 3C_2 x^* + 10C_3 X^3, \]

\[ \beta_3 = C_2 + 10C_3 X^2, \quad \beta_4 = 5C_3 X^*, \quad \beta_5 = C_3. \]  

(10)

With the condition \( A \ll 1 \), we assume \( |Y| \ll 1 \), and neglect the nonlinear term in Eq. (9). Then in the limit \( t \to \infty \), \( Y(t) = A_L \cos(\omega t - \phi) \), where

\[ A_L = \frac{A}{\sqrt{\omega^4 + \omega^2\gamma^2}} \phi = \arctan \left( \frac{\omega \gamma}{\omega_r^2} \right), \]

(11)

and the resonance frequency is \( \omega_r = \sqrt{C_1} \), when the slow motion takes place around the equilibrium point \( X^* = 0 \); the other \( \omega_r = \sqrt{C_3} \). The resonance amplitude \( Q \) is defined as

\[ Q = \frac{A_L}{A} = \frac{1}{\sqrt{\omega^4 + \omega^2\gamma^2}}. \]

(12)

Going farther, we can analyze the VR according to Eq. (12) from the theoretical point of view. It is emphasized that in Figs. 2–3 we only take \( \omega_r = \sqrt{C_1} \), and do not discuss the case \( \omega_r = \sqrt{C_3} \). That is because the results are similar to the case which the resonance takes place around \( X_1^* = 0 \). Taking \( \omega_r = \sqrt{C_1} \) into Eq. (12), we can obtain the expression of the response amplitude \( Q \), and it is as follow:

\[ Q = \frac{1}{\sqrt{\omega^4 + \omega^2\gamma^2}} = \frac{1}{\sqrt{\left( \frac{-12bcB^2}{\gamma^2\Omega^2} + 2c^2 + \frac{45b^2B^4}{4\gamma^4\Omega^2} \right)^2 + \omega^2\gamma^2}}. \]

(13)
FIG. 2: The response amplitude $Q$ of this system with an additive low-frequency signal and a high-frequency signal is plotted as a function of the amplitude $B$ of the high-frequency signal: (a) $\gamma = 1$ and $\omega = 0.1$ with $\Omega = 5, 10$; (b) $\Omega = 5$ and $\omega = 0.1$ with $\gamma = 0.5$, and 1; (c) $\gamma = 1$ and $\Omega = 5$ with $\omega = 0.1$ and 0.5. The other parameters are $b = 0.1, c = 1$.

From the theoretical expression of $Q$, we can determine the positions at which the VR occurs. $Q$ depends on the parameters $B$, $\Omega$, $\gamma$, and $\omega$. We can rewrite Eq. (13) as $Q = 1/\sqrt{S}$ where

$$S = \left( -\frac{12bcB^2}{\gamma^2\Omega^2} + 2c^2 + \frac{45b^2B^4}{4\gamma^4\Omega^4} \right)^2 + \omega^2\gamma^2. \quad (14)$$

If the function $S$ is a minimum, a local minimum of $S$ represents a resonance. Therefore, by finding the minima of $S$, the positions at which the resonance occurs can be determined.

Firstly, the relation between $Q$ and $B$ are revealed in Fig. 2. The roots of $\frac{d^2S}{dB^2} = 0$ with $\frac{d^2S}{dB^2} > 0$ are $B_1 = \frac{1}{15}\gamma\sqrt{30}\sqrt{\frac{(1-\sqrt{6})}{b}}\Omega$, and $B_2 = \frac{1}{15}\gamma\sqrt{30}\sqrt{\frac{(1+\sqrt{6})}{b}}\Omega$, which are the positions at which the resonance occurs. There exists two maxima, which is the double VR phenomenon. It is found that the positions are determined by $\Omega$ and $\gamma$, and are independent of the frequency of the low-frequency signal $\omega$. From the expressions of $B_1$ and $B_2$, we find that the positions at which the response occurs shift from the smaller value to the larger
FIG. 3: The response amplitude $Q$ this system with an additive low-frequency signal and a high-frequency signal is plotted as a function of the frequency $\Omega$ of the high-frequency signal: (a) $\gamma = 1$ and $B = 5$ with $\omega = 0.1$, 0.5; (b) $B = 5$ and $\omega = 0.1$ with $\gamma = 0.5$, and 1; (c) $\gamma = 1$ and $\omega = 0.1$ with $B = 5$ and 10. The other parameters are $b = 0.1$, $c = 1$.

value with $\Omega$ increasing. For example, taking $b = 0.1$, $c = 1$, $\Omega = 10$, and $\gamma = 1$, we get $B_1 = 14.378$ and $B_2 = 29.325$ (see Fig. 2a, the dashed represents $\Omega = 10$), and taking $b = 0.1$, $c = 1$, $\Omega = 5$, and $\gamma = 1$, we get $B_1 = 7.188$ and $B_2 = 14.662$ (see Fig. 2a, the real line represents $\Omega = 5$). It is emphasized that the height of the peaks has no distinct difference for different $\Omega$. However, with $\gamma$ increasing, it is found that the maxima of $Q$ become small, and the positions at which the response occurs shift from the smaller value to the larger value (see Fig. 2b). That is to say, the VR may disappear when $\gamma$ reaches a critical value; this means that VR is restrained by the dissipation constant $\gamma$. Figure 2c shows that the positions where the resonance occurs are independent of $\omega$, only the maxima of $Q$ decrease as $\omega$ increases. We can also draw the result from Eq. (13), when $\omega$ is a maximum, $Q$ reaches a minimum. That is to say, there is a critical $\omega_c$. When $\omega > \omega_c$, the resonance will not occur.

Secondly, Figure 3 shows the relation between $Q$ and $\Omega$ for different parameters. We can get the positions at which the resonance occurs in the $Q - \Omega$ plot. The positions are as follows: $\Omega_1 = \frac{\sqrt{2}}{2} \sqrt{\frac{b(4-\sqrt{6})}{c}}B\gamma^{-1}$ and $\Omega_2 = \frac{\sqrt{2}}{2} \sqrt{\frac{b(4+\sqrt{6})}{c}}B\gamma^{-1}$. The resonance positions are
determined by $B$ and $\gamma$ and, are independent of the frequency of the low-frequency signal $\omega$. With $B$ increasing, the positions of the two peaks shift from the smaller $\Omega$ to larger $\Omega$ (see Fig. 3c) and the height does not change. However, the position and the height of the two peaks are affected by $\gamma$. As $\gamma$ increases, the positions of the two peaks shift from the larger $\Omega$ to smaller $\Omega$, and the height becomes low (see Fig. 3b), which is opposite to the effect of $\gamma$ in the $Q - B$ plot. $\omega$ has no effect on the position at which the resonance occurs, but can restrain the height of the two peaks. It is like the case in Fig. 2c.

III. NUMERICAL RESULTS

In order to check the validity of the approximate method, we define the response amplitude $Q$ for describing the signal transmission in the system at the lower frequency $\omega$, which is given by [1]
FIG. 5: Numerical simulations of $Q$ according to Eq. (15) is plotted as functions of $\Omega$. (a) $\gamma = 1$ and $B = 5$ with $\omega = 0.1, 0.5$; (b) $B = 5$ and $\omega = 0.1$ with $\gamma = 0.5, 1$; (c) $\gamma = 1$ and $\omega = 0.1$ with $B = 5$ and 10. The other parameters are $b = 0.1, c = 1$.

$Q = \frac{\sqrt{B_s^2 + B_c^2}}{A}$, \hspace{1cm} (15)

with

$B_s = \frac{2}{nT} \int_0^{nT} x(t) \sin(\omega t) dt$, \hspace{1cm} (16)

and

$B_c = \frac{2}{nT} \int_0^{nT} x(t) \cos(\omega t) dt$, \hspace{1cm} (17)

where $T = \frac{2\pi}{\omega}$ and $n$ is a positive integer.

We calculate the response amplitude $Q$ numerically with the fourth-order Runge-Kutta method with fixed step sizes $\Delta t = 0.01$. The initial condition $x(0)$ is chosen randomly, and the total time is $t = 2 \times 10^6$. The results are plotted in Figs. 4–5 with the same parameters as in Figs. 2–3. The numerical results are found to be in good agreement with the theoretical predictions.
IV. CONCLUSIONS

In conclusion, the occurrence of VR in an overdamped oscillator with the triple-well potential driven by a low-frequency force and a high-frequency force is investigated. Making use of the method of direct separation of the motion, we obtain an approximate theoretical expression for the response amplitude $Q$. From the analytical expression of $Q$ we determined the values of $B$ and $\Omega$ at which VR occurs. The numerical results are found to be in good agreement with the theoretical predictions. Since the high-frequency signal input is deterministic and is more controllable, we expect that the scheme could be of great significance for potential applications due to its simplicity and high efficiency.

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