The Variances in a Single-Mode Superposed Squeezed State

A. S. Daoud

Department of Mathematics, Faculty of Sciences, University of Zagazig, Zagazig, Egypt
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We find explicit expressions for the variances in a single-mode superposed squeezed state as a function of a certain parameter. Formally, the parameter is given as a function of the mean free path (which can be measured experimentally) of the atoms of a material medium through which the natural light passes. Physically, this parameter is just the mean number of photons present in the coherent state of the mode under investigation. Some interesting new results follow as a consequence of using the P-representation of the density operator for the squeezed state of the mode.

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I. INTRODUCTION

The calculation of the variances in a quantum state leads to the determination of the total noise of that state. The knowledge of the noise level of a state is essential to estimate the value of such a state in practice. According to Schumaker [1], the variances in a single-mode state are defined as the mean-square uncertainties in the real and imaginary parts of the annihilation operator of the mode. Hence the total noise of the state is given by the sum of the variances in that state.

In a beam of squeezed light, the quantum noise level could be reduced below the zero-point fluctuations that minimize the uncertainty product given by Heisenberg’s uncertainty relation. The quantum fluctuations in a coherent state are equal to the zero-point fluctuations and are randomly distributed in phase. Even an ideal laser operating in a pure coherent state would still possess quantum noise, due to zero-point fluctuations [2]. In optical communication systems which use a coherent beam of laser light propagating in optical fibers, the ultimate limit to the noise is given by the quantum noise or zero-point fluctuations. If, instead, a beam of squeezed light were used to transmit information in the quadrature phase that had reduced fluctuations, the quantum noise level could be reduced below the zero-point fluctuations. Yuen and Shapiro [3, 4] propose optical communication systems based on light signals with phase sensitive quantum noise.

Indeed, when the density operator for a squeezed pure state represents the state of the mode, the variances are still functions of the squeeze factor, which varies from zero to infinity. In addition, the variances in such states are independent of the field amplitude of the coherent state. As well, the product of the uncertainties is still equal to the minimum allowed value, as in the coherent state. On the other hand, according to Schumaker [1], the sum of the variances in a state must be equal to the number of photons present in the mode.
state including the contribution of the ground state of the mode. In the case of the pure state, this statement cannot be realized. In fact all these circumstances will disappear, as we shall see in the present research paper, if the mode is represented by a mixed state.

Here, the quantum state of the mode is considered as the P-representation of the density operator for a single-mode squeezed state. The weighing factor of this representation may be factored into two independent functions. One factor takes the form of Glauber-Sudarshan’s P-function for coherent states, while the other is a new function that depends only on the squeeze factor and squeeze angle of the state.

II. DEFINITIONS AND NOTATIONS

II-1. Single-mode coherent state

Let \( a^+ (a) \) be the creation (annihilation) operator in (off) a mode of the radiation field. For each mode of the field, the displacement operator \( D(\alpha) \) [5–8] takes the form

\[
D(\alpha) = \exp[\alpha a^+ - \alpha^* a],
\]

with \( \alpha \) being a complex number, which may be written as

\[
\alpha = \sigma e^{i\theta},
\]

where the real numbers \( \sigma \) and \( \theta \) are the amplitude and phase of \( \alpha \). These numbers are restricted to the range

\[
0 \leq \sigma < \infty, \quad 0 < \theta \leq 2\pi.
\]

Since each \( a \) and \( a^+ \) is noncommuting with the operator \( \alpha a^+ - \alpha^* a \), we can easily show that the displacement operator (1) provides the following operator relations

\[
D^+ (\alpha) a D(\alpha) = a + \alpha,
\]

\[
D^+ (\alpha) a^+ D(\alpha) = a^+ + \alpha^*.
\]

The additive nature of the transformation relation (4) implies that when the displacement operator acts on a state it changes all moments of \( a \) and \( a^+ \).

The single-mode coherent state, symbolized by the state vector \( |\alpha\rangle \), may be defined as the state unitarily related to the ground state \( |0\rangle \) of the oscillator by

\[
|\alpha\rangle = D(\alpha)|0\rangle.
\]

This state is an eigenstate of the annihilation operator with eigenvalue \( \alpha \), namely

\[
a|\alpha\rangle = \alpha|\alpha\rangle.
\]

Glauber [5, 6] and others [9, 10], beginning in the early 1960s, have used coherent states to build a powerful description of the electromagnetic field.
II-2. Single-mode squeezed states

The single-mode squeeze operator \([11–15]\) is defined by

\[
S(r\varphi) = \exp \left[ \frac{1}{2} z^* a^2 - \frac{1}{2} z (a^+)^2 \right],
\]

where the complex factor \(z\) has the form

\[
z = re^{2i\varphi}.
\]

The real numbers \(r\) and \(\varphi\) are called, respectively, the squeeze factor and the squeeze angle of the squeezed state. These numbers are defined so that

\[
0 \leq r < \infty, \quad -\frac{\pi}{2} < \varphi \leq \frac{\pi}{2}.
\]

Since each \(a\) and \(a^+\) is noncommuting with the operator \(\exp(-2i\varphi)a^2 - \exp(2i\varphi)(a^+)^2\), it is easy to show that

\[
S^+(z)aS(z) = a \cosh r - a^+ \exp(2i\varphi) \sinh r,
\]

\[
S^+(z)a^+S(z) = a^+ \cosh r - a \exp(-2i\varphi) \sinh r.
\]

These relations show that the squeeze operator (7) mixes \(a\) and \(a^+\). Therefore, it induces a correlation between the position and the momentum variables.

A single-mode squeezed state \([1, 2, 11–15]\), symbolized by \(|\alpha, z\rangle\), is defined as the displacement operator (1) acting on the squeezed vacuum \(S(z)|0\rangle\), i.e.

\[
|\alpha, z\rangle = D(\alpha)S(z)|0\rangle.
\]

Under the name of “two photon coherent state” Yuen [13] discussed these states in detail. The properties of the squeezed states as well as its possible applications in practice have been considered by Hollenhorst [14].

II-3. Dependence of the mean number of photons on the mean free path

Let \(\Omega\) be a unit vector coinciding with the direction of the velocity of a particle when its energy is \(u\). We introduce the function \(g(\mu_0, u - u')\) to represent the relative probability of a particle being left with the pair \((\Omega, u)\) as a result of a collision, before which it was characterized by the pair \((\Omega', u')\) \((\mu_0 = \Omega \cdot \Omega'\) being the cosine of the angle through which the particle is scattered). Next assume that \(g(\mu_0, u - u')\) may be expanded in terms of the Legendre polynomials \(P_j(\mu_0)\):

\[
g(\mu_0, u - u') = \frac{1}{4\pi} \sum_{j=0}^{\infty} (2j + 1)g_j(u - u')P_j(\mu_0),
\]

with

\[
g_j(u - u') = 2\pi \int_{-1}^{+1} d\mu_0 g(\mu_0, u - u')P_j(\mu_0),
\]
where we have used the normalization condition of Legendre polynomials. Now let us define the numbers $A$ and $B$ as follows:

\[ A = 2\pi \int_{u-\gamma}^{u} du' (u - u') g_0(u - u') , \]  

\[ B = 2\pi \int_{u-\gamma}^{u} du' g_1(u - u') , \]

where $g_0(u - u')$ and $g_1(u - u')$ are obtained from (13) for \( j = 0, 1 \) respectively, while $\gamma$ represents the maximum energy loss (the maximum energy loss occurs when the particle is scattered through an angle of 180°). The average of the photons (symbolized by $\varepsilon$) occupying a thermal mode as a function of the mean free path, $T(u)$, of the atoms of a medium through which the natural light passes takes the following form [16–18]:

\[ \varepsilon = \frac{4}{3A(1 - B)} \int_{0}^{u} T^2(u') du' . \]  

Although relation (16) is simple, and it can be obtained as a result of applying a very crude model of the interaction between light and atoms, it is the basis on which many phenomena characterizing the statistical properties of light can be built. As well, owing to relation (16), one can measure experimentally the average number of photons occupying a thermal mode by measuring the mean free path $T$.

### III. THE DENSITY OPERATOR FOR A SINGLE-MODE SQUEEZED STATE

The density operator for the pure squeezed state $|\alpha, z\rangle$ is just the following projection operator [5, 8, 19–21]:

\[ \tilde{\rho} = |\alpha, z\rangle \langle z, \alpha| , \]  

where $|\alpha, z\rangle$ is defined by (11). In this case, one assumes that we have complete knowledge about the state of the system we are studying. Therefore, we can say, with certainty that every element of the system is in the state $|\alpha, z\rangle$. Now, if we insert (11) and its adjoint into (17) then we obtain

\[ \tilde{\rho} = D(\alpha) S(z) |0\rangle \langle 0| S^+(z) D^+(\alpha) , \]  

where $S^+$ and $D^+$ are the adjoints of $S$ and $D$, respectively. Since $|0\rangle \langle 0| = 1$ and each of $S$ and $D$ is a unitary operator, we can easily see that

\[ \text{Tr} \tilde{\rho} = 1 . \]  

The mean photon number in the state (18) is [1, 2]

\[ \langle \tilde{n}(\varepsilon) \rangle = \sigma^2 + \sinh^2 r . \]
Of course the mean number \( (20) \) is equal to the expectation value of the number operator \( a^+a \), with reference to state \( (18) \).

If we do not have complete knowledge about the state of the system we are studying, then we assume that the probability that it is in the state \( |\alpha, z\rangle \) is \( W(\alpha, z) \) (say). In this case the density operator, symbolized by \( \rho \), is just the superposition of the operators \( (17) \) [5, 8, 18], namely,

\[
\rho = \int \frac{d2\alpha}{\pi} \int d^2z \ W(\alpha, z) |\alpha, z\rangle \langle z, \alpha|.
\]

This defines the P-representation of the density operator for a single-mode squeezed state. According to \((2)\) and \((8)\), the differential elements \( d^2\alpha \) \( (=d(\text{Re} \alpha)d(\text{Im} \alpha)) \) and \( d^2z \) \( (=d(\text{Re} z)d(\text{Im} z)) \) can be written again in the forms

\[
d^2\alpha = \sigma d\sigma d\theta,
\]

and

\[
d^2z = r dr d\varphi .
\]

As a density operator \( \rho \) must be Hermitian, and its trace is equal to 1. This leads to the following constraint on \( W(\alpha, z) \):

\[
\int \frac{d2\alpha}{\pi} \int d^2z \ W(\alpha, z) = 1,
\]

where the integrals run over intervals \((3)\) and \((9)\) with respect to \( \alpha \) and \( z \), while \( d^2\alpha \) and \( d^2z \) are defined by \((22)\) and \((23)\).

In terms of \( \sigma \), \( r \), and \( \varphi \) (see transformations \((2)\) and \((8)\)) the weighing factor \( W(\alpha, z) \) takes the form \([18]\]

\[
W(\alpha, z) = 2(\pi^3 \varepsilon^{-5})^{-1/2} r \cos \varphi \exp \left[ -\left( \sigma^2 + r^2 \right) / \varepsilon \right],
\]

where \( \varepsilon \), defined by \((16)\), represents the mean number of photons occupying the coherent state of the mode. In fact the function \((25)\) can be factored into two independent functions: The first, symbolized by \( P(\alpha) \), takes the form of the Glauber-Sudarshan \( P \) function \([5, 8]\) for a single-mode coherent state, i.e.

\[
P(\alpha) = \frac{1}{\pi \varepsilon} \exp(-\sigma^2 / \varepsilon).
\]

In fact it is easy to show that

\[
\int \frac{d2\sigma}{\pi} \exp(-\sigma^2 / \varepsilon) \int_0^\infty d\sigma \int_0^{2\pi} d\theta \int_0^\infty \exp(-\sigma^2 / \varepsilon) d\sigma = 1.
\]

This means that \( P(\alpha) \) is normalized to unity on the intervals \((3)\). Indeed, it is clear that \( P(\alpha) \) is positive everywhere in the plane and takes the same form as the probability distribution for the total displacement, which results from a random walk \([5, 8, 22]\).
The second factor, symbolized by \( Q(z) \), is [17, 18]
\[
Q(z) = 2r(\pi z^3)^{-1/2} \cos \varphi \exp(-r^2/\varepsilon).
\] (28)

As well, we can show that \( Q(z) \) is normalized over the intervals (9) to unity, namely,
\[
\int \int Q(z) d^2 z = 2(\pi z^3)^{-1/2} \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi \int_0^\infty r^2 \exp(-r^2/\varepsilon) dr = 1.
\] (29)

The function \( Q(z) \) defined by (28) is positive for all values of \( r \) on the segment \(-\pi/2 < \varphi < \pi/2\), while it is negative otherwise. This function is distributed in phase angle through its dependence on \( \cos \varphi \). Thus, \( Q(z) \) cannot be considered as a probability density, but it is a real-valued weight function. The above discussion shows that the total weight function \( W(\alpha, z) = P(\alpha)Q(z) \), defined by (25), is positive on the right-half plane while it is negative on the left-half plane, and it is distributed in the phase angle \( \varphi \).

Taking into account that the squeezed state \( |\alpha, z\rangle \) is normalized to unity such that \( \langle z, \alpha |\alpha, z\rangle = 1 \), constraints (27) and (29) imply that the total weight factor \( W(\alpha, z) \) fulfills the normalization condition (24). Indeed, the mean number of photons in the state (21) is given by [17, 18]
\[
\langle n(\varepsilon) \rangle = \varepsilon + (\varepsilon + \frac{1}{2}) \exp(\varepsilon) - \frac{1}{2},
\] (30)
where \( \varepsilon \) is defined by (16).

In fact expression (21) is the most general P-representation of the density operator for a single-mode radiation field. If we replace \( |\alpha, z\rangle \langle z, \alpha| \) by \( |\alpha\rangle \langle \alpha| \), where \( |\alpha\rangle \) is the coherent state of the mode, then (21) reduces to the P-representation for a single-mode coherent state [8, 9], namely
\[
\rho_c = \int \int P(\alpha)|\alpha\rangle \langle \alpha| d^2 \alpha,
\]
where the constraint (29) has been applied. As well, if \( |\alpha, z\rangle \langle z, \alpha| \) is replaced by \( |z\rangle \langle z| \) where \( |z\rangle \) denotes the squeezed vacuum state of the mode, then, by means of (27), the operator (21) takes the form
\[
\rho_{sv} = \int \int Q(z) |z\rangle \langle z| d^2 z.
\]
This is the P-representation of the density operator for the squeezed vacuum state of the mode. In fact an expansion of \( \rho_{sv} \) in terms of the number states of the mode yields some new interesting statistical properties for the field (see [17, 18]).

IV. THE QUADRA TURES COMPONENTS OF THE ELECTRIC FIELD

With time \( t \) and frequency \( \omega \), the electric field may be written as [2]
\[
E(t) = C(a e^{-i\omega t} + a^+ e^{i\omega t}),
\] (31)
where $C$ is a constant including the spatial wave function. In the quantum theory of radiation the amplitudes $a$ and $a^+$ are quantum mechanical operators which obey boson commutation relations. Operators $a$ and $a^+$ are called, respectively, the annihilation and creation operators. Since $a$ and $a^+$ are not Hermitian, they may be complex operators. Thus, we can write $a$ as a complex operator such as

$$a = \xi + i\zeta,$$

(32)

where $\xi$ and $\zeta$ are Hermitian operators. In terms of $\xi$ and $\zeta$, one may write (31) as

$$E(t) = C(\xi \cos \omega t + \zeta \sin \omega t).$$

(33)

Thus $\xi$ and $\zeta$ may be identified as the amplitudes of the two quadrature phases of the field. Consider now the rotated complex amplitudes

$$q + ip = (\xi + i\zeta)e^{-i\psi},$$

(34)

where $\psi$ is a rotational angle. Solving (34) with respect to $q$ and $p$, one obtains

$$q = \frac{1}{2}(ae^{-i\psi} + a^+e^{i\psi}),$$

(35)

and

$$p = -\frac{i}{2}(ae^{-i\psi} - a^+e^{i\psi}),$$

(36)

where (32) has been used. Since $[a, a^+] = 1$, it is easy to see that $q$ and $p$ obey the commutation relation

$$[q, p] = \frac{1}{2}. $$

(37)

By means of (37), the Heisenberg uncertainty principle enables us to have the following relation for the uncertainties in $q$ and $p$:

$$\Delta q \Delta p \geq \frac{1}{4}. $$

(38)

A family of minimum uncertainty states is defined by taking the equals sign. One such class of minimum uncertainty states is the coherent state.

V. THE VARIANCES IN A PURE SQUEEZED STATE

The variances in a state are defined as the mean-square fluctuations in the real and imaginary parts of the annihilation operator $a$, evaluated with reference to that state [1, 9, 10]. Accordingly, the variances in the pure squeezed state (18) are the squared uncertainties in the amplitudes $\xi$ and $\zeta$ defined by (32). Equivalently, these variances are the squared


uncertainties in \( q \) and \( p \) defined by (35) and (36), which are related to \( \xi \) and \( \zeta \) through relation (34), which differs from (32) by the multiplicative phase factor \( \exp(-i\psi) \).

Now, if we take into account the unitary nature of the operators (1) and (7), we may obtain the following expectation values evaluated relative to state (18):

\[
\langle z, \alpha | q^2 | \alpha, z \rangle = \frac{1}{2} \sigma^2 + \frac{1}{2} \text{Re}(\alpha^2 e^{-2i\varphi}) + \frac{1}{4} e^{-2r}, \tag{39}
\]

and

\[
\langle z, \alpha | q | \alpha, z \rangle = \text{Re}(\alpha e^{-i\varphi}), \tag{40}
\]

where we have used the operator relations (4) and (10) as well as the completeness of the number states. Thus, one variance in the pure squeezed state (17) follows by subtracting the square of (40) from (39). It takes, after a little manipulation, the simple form

\[
\tilde{V}(q) = \frac{1}{4} e^{-2r}. \tag{41}
\]

Similarly, for \( p \) defined by (36) we can get the following averages, evaluated with respect to state (18):

\[
\langle z, \alpha | p^2 | \alpha, z \rangle = \frac{1}{2} \sigma^2 - \frac{1}{2} \text{Re}(\alpha^2 e^{-2i\varphi}) + \frac{1}{4} e^{2r}, \tag{42}
\]

and

\[
\langle z, \alpha | p | \alpha, z \rangle = \text{Re}(i\alpha^* e^{i\varphi}). \tag{43}
\]

Hence, the other variance in state (18) results similarly by subtracting the square of (43) from (42):

\[
\tilde{V}(p) = \frac{1}{4} e^{2r}. \tag{44}
\]

In reality, relations (39)–(44) are valid only for \( \psi = \varphi \). From (41) and (44), it is clear that for a single-mode squeezed pure state the variance of \( q \) is minimized and is a factor \( e^{-2r} \) smaller than its coherent-state value, while the variance of \( p \) is maximized and is a factor \( e^{2r} \) larger than its coherent-state value.

Functions (41) and (44) give the variances in the squeezed state \(| \alpha, z \rangle \) represented by the pure-state density operator (18). Clearly these variances are independent of the field amplitude \( \alpha \). This means that the associated coherent state does not contribute to the variances in the state. In fact the variances in the squeezed vacuum pure state \(| z \rangle = S(z)|0 \rangle \), which is already independent of \( \alpha \), take also the forms (41) and (44) (see Ref. [23]). As we shall see later, the contribution of the associated coherent state to the variances in a squeezed state will appear when we consider the mixed states instead of pure states.
According to (41) and (44), we can easily deduce the following relation for the uncertainties \( \Delta q (= |V(q)|^{1/2}) \) and \( \Delta p (= |V(p)|^{1/2}) \) in \( q \) and \( p \):

\[
\Delta q \Delta p = \frac{1}{4},
\]

which is still the minimum value allowed by quantum mechanics theory, just as in the coherent states. Indeed, if we consider the mixed states instead of the pure states, then we shall obtain two different values for the product \( \Delta q \Delta p \) in the two cases.

VI. THE VARIANCES IN A SQUEEZED MIXED-STATE

Recall that the variances in a state are defined as the mean-square fluctuations in the real and imaginary parts of the annihilation operator, evaluated with reference to that state. Accordingly, one variance in the squeezed state \( |\alpha, z \rangle \) is given by

\[
V(q) = \text{Tr}(\rho q^2) - [\text{Tr}(\rho q)]^2,
\]

where \( \rho \) and \( q \) are, respectively, the density operator (21) and the quadrature (35), while the symbol \( \text{Tr} \) stands for the trace. By means of (21), \( \text{Tr}(\rho q) \) is

\[
\text{Tr}(\rho q) = \iint W(\alpha, z) \langle z, \alpha | q | \alpha, z \rangle d^2 \alpha d^2 z.
\]

Inserting (40) and the factored form of \( W(\alpha, z) (= P(\alpha)Q(z)) \) into the right side of the above relation, we obtain

\[
\text{Tr}(\rho q) = \iint Q(z) d^2 z \iint P(\alpha) \text{Re}(\alpha e^{-i\varphi}) d^2 \alpha.
\]

Now, if we substitute (2), (22), and (26) into the right side of the above relation and carry out the integrals over intervals (3) then it is easy to show that

\[
\text{Tr}(\rho q) = 0.
\]

(46)

Similarly, \( \text{Tr}(\rho q^2) \) is given by

\[
\text{Tr}(\rho q^2) = \iint W(\alpha, z) \langle z, \alpha | q^2 | \alpha, z \rangle d^2 \alpha d^2 z.
\]

(47)

By means of the factored form of \( W(\alpha, z) \) and the matrix element (39), the above trace becomes

\[
\text{Tr}(\rho q^2) = \frac{1}{2} \iint Q(z) d^2 z \iint P(z) \sigma^2 d^2 \alpha + \frac{1}{2} \iint P(\alpha)Q(z) \text{Re}(\alpha^2 e^{-2i\varphi}) d^2 \alpha d^2 z
\]

\[
+ \frac{1}{4} \iint P(\alpha) d^2 \alpha \iint Q(z) \exp(-2r) d^2 z.
\]
If we use the constraints (27) and (29), then \( \text{Tr}(\rho q^2) \) reduces to
\[
\text{Tr}(\rho q^2) = \frac{1}{2} \iint P(\alpha)\sigma^2 d^2\alpha + \frac{1}{2} \iint P(\alpha)Q(z)\text{Re}(\alpha^2 e^{-2i\sigma})d^2\alpha d^2z + \frac{1}{4} \iint Q(z)\exp(-2r)d^2z.
\]

It is clear that the integrand in the first term on the right depends on the amplitude \( \alpha \) alone, and the integrand in the third depends on the amplitude \( z \) alone, while the integrand in the second depends on both \( \alpha \) and \( z \). Thus the first term represents the contribution of the coherent state, and the third represents the contribution of the squeezed vacuum state, while the second represents the overlap between the two states.

Now, since \( \int_0^{2\pi} \exp(\pm i\theta)d\theta = 0 \), the second integral on the right of the above relation will vanish over the intervals (3). If we take into account this fact and use the functions (26) and (28), then we get
\[
\text{Tr}(\rho q^2) = \frac{1}{2\pi \varepsilon} \int_0^{2\pi} d\theta \int_0^\infty \sigma^3 \exp(-\sigma^2/\varepsilon)d\sigma \\
+ \frac{1}{2}(\pi \varepsilon^3)^{-1/2} \int_{-\pi/2}^{\pi/2} \cos \phi d\phi \int_0^\infty \int_0^\infty r^2 \exp[-(r^2 + 2\varepsilon r)/\varepsilon]dr,
\]
where (2), (3), (8), and (9), as well as the elements of area (22) and (23), have been used. If we carry out the integrals, the first term yields \( \frac{1}{2\varepsilon} \), which gives the contribution of the coherent state, while the contribution of the squeezed vacuum state comes from the second term and assumes the value \( [(2\varepsilon^2 + 1)\nu(\varepsilon) - \varepsilon^2]/(2\pi \varepsilon^3)^{1/2} \). Therefore, \( \text{Tr}(\rho q^2) \) takes the following form:
\[
\text{Tr}(\rho q^2) = \frac{1}{2\varepsilon} + \frac{1}{2}(\pi \varepsilon)^{-1/2}[(2\varepsilon + 1)\nu(\varepsilon) - \varepsilon],
\]
with
\[
\nu(\varepsilon) = \int_0^\infty \exp[-(r^2 + 2r \varepsilon)/\varepsilon]dr.
\]

According to (46) and (48), the variance \( V(q) \) takes the form
\[
V(q) = \frac{1}{2\varepsilon} + \frac{1}{2}(\pi \varepsilon)^{-1/2}[(2\varepsilon + 1)\nu(\varepsilon) - \varepsilon].
\]

Now let us focus our attention on the squared uncertainty in the quadrature \( p \) defined by (36). In fact, for the same reasons and by typical techniques, the other variance in the squeezed state is given by
\[
V(p) = \frac{1}{2} \iint P(\alpha)\sigma^2 d^2\alpha + \frac{1}{4} \iint Q(z)\exp(2r)d^2z,
\]
where we have used the factored form of \( W(\alpha, z) \) as well as relations (42) and (43). By means of the identity \( e^{2r} = 2 \cosh 2r - e^{-2r} \), the variance \( V(p) \) becomes
\[
V(p) = \frac{1}{2} \iint P(\alpha)\sigma^2 d^2\alpha + \frac{1}{4} \iint Q(z)[2 \cosh 2r - e^{-2r}]d^2z.
\]
Similarly, since the integrand in the first term on the right depends on \( \alpha \) alone, it represents the contribution of the coherent state, and, since the integrand in the second term depends on \( z \) alone, it gives the contribution of the squeezed vacuum state. If we carry out the integrals over the intervals (3) and (9), then we get

\[
V(p) = \frac{1}{2} \varepsilon + (\varepsilon + \frac{1}{2}) e^\varepsilon - \frac{1}{2} (\pi \varepsilon)^{-1/2}[(2\varepsilon + 1) \nu(\varepsilon) - \varepsilon],
\]

(51)

where (22), (23), (26), and (28) have been used, while \( \nu(\varepsilon) \) is defined by (49). An asymptotic expansion of \( \nu(\varepsilon) \) is obtained in the appendix, the result is \( \nu(\varepsilon) \sim -1/2 \), which permits us to write (48) and (51) as

\[
V(q) = \frac{1}{2} \varepsilon + \frac{1}{4} (\pi \varepsilon)^{-1/2}
\]

(52)

and

\[
V(p) = \frac{1}{2} \varepsilon + (\varepsilon + \frac{1}{2}) e^\varepsilon - \frac{1}{4} (\pi \varepsilon)^{-1/2}.
\]

(53)

The first term on the right of (52) and of (53) constitutes the contribution of the amplitude \( \alpha \) to the variances in the state. Obviously, this contribution distributes equally between the variances \( V(q) \) and \( V(p) \). In contrast, the amplitude \( z \) makes a smaller contribution to the variance of the quadrature of \( q \) at the expense of an increased contribution to the variance of the quadrature of \( p \).

Relations (52) and (53) give the variances in the single-mode squeezed state represented by the density operator (21). These variances are dependent exclusively on the number \( \varepsilon \). Physically, this number is the mean number of photons present in the coherent state of the mode. Formally, \( \varepsilon \) is given by the integral (16).

VII. CONCLUDING REMARKS

According to Schumaker [1], the total noise of a state, or equivalently the sum of the variances in a state, can be thought of as the noise content of the state in units of photon number; it is the number of photons, including a half quantum due to the zero-point noise, that would be left in the state if the mean excitations were removed. Here, we shall conclude that only the mixed state (21) verifies this statement. The sum of \( V(q) \) and \( V(p) \), defined by (52) and (53), can be written as

\[
V(q) + V(p) = \langle n(\varepsilon) \rangle + \frac{1}{2},
\]

where \( \langle n(\varepsilon) \rangle \), given by (30), is the mean number of photons present in state (21). The half in the right side is the contribution of the ground state of the mode that would be left in the state if the mean excitation were removed. Hence, the sum of variances in the mixed state (21) is equal to the total number of photons in the state, including the contribution of the ground state of the mode.
On the other hand, the sum of (41) and (44) gives the total variances in the pure state (17), that is
\[ \tilde{V}(q) + \tilde{V}(p) = \frac{1}{2} \cosh 2r, \]
which can be written again in the following form:
\[ \tilde{V}(q) + \tilde{V}(p) = \sinh^2 r + \frac{1}{2}. \]
It is clear that the right side of the above relation is completely different from that of relation (20), which gives the mean number of photons in the pure state (17). Therefore, we conclude that the sum of the variances in the pure state is not equal to the total number of photons including the contribution of the ground state of the mode. This is contrary to Schumaker’s statement.

In contrast with the pure state, from (52) one cannot conclude that \( V(q) \) is less than 1/4, which is the corresponding variance in a coherent state. As well, it is clear from (52) and (53) that \( \Delta q \Delta p \) is not equal to 1/4 as in the pure state. Thus, the consideration only of the pure states leads to incomplete results.

**APPENDIX A: ASYMPTOTIC EXPANSION OF \( \nu(\varepsilon) \)**

The number \( \nu(\varepsilon) \), defined by (49), may be written as
\[ \nu(\varepsilon) = e^\varepsilon \int_0^\infty \exp(-s^2/\varepsilon) ds, \]
which can easily reduce to the following form
\[ \nu(\varepsilon) = e^{\varepsilon \sqrt{\frac{\pi \varepsilon}{2}}} [1 - \Phi(\sqrt{\varepsilon})], \tag{A1} \]
where
\[ \Phi(\sqrt{\varepsilon}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\varepsilon}} \exp(-x^2) dx. \tag{A2} \]
Now, from the fact that \( \int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi} \), it follows immediately that the limit of \( \Phi \), as \( \varepsilon \) tends to \( \infty \), is equal to 1. Relation (A2) may be written again in the form
\[ 1 - \Phi(\sqrt{\varepsilon}) = \frac{2}{\sqrt{\pi}} \int_{\sqrt{\varepsilon}}^{\infty} \exp(-x^2) dx. \]
Integrating by parts gives
\[ 1 - \Phi(\sqrt{\varepsilon}) = \frac{1}{\sqrt{\pi}} \left[ e^{-1/2} \exp(-\varepsilon) - \int_{\sqrt{\varepsilon}}^{\infty} x^{-2} \exp(-x^2) dx \right]. \]
Iteration of the integration by parts soon yields
\[
[1 - \Phi(\sqrt{\varepsilon})] \exp(\varepsilon) = \frac{1}{\pi} \sum_{k=0}^{n} \left[ (-1)^{k} \Gamma(k + \frac{1}{2}) / \varepsilon^{k + 1/2} \right] \\
+ \frac{2}{\pi} (-1)^{n+1} \Gamma(n + \frac{3}{2}) \exp(\varepsilon) \int_{\sqrt{\varepsilon}}^{\infty} x^{-2n-2} \exp(-x^2) dx ,
\]
where \( \Gamma \) denotes the Gamma function. The variable of integration is never less than \( \sqrt{\varepsilon} \).

We can replace the factor \( x^{-2n-2} \) in the integrand by \( x\varepsilon^{-n-3/2} \) and thus obtain
\[
\left| 1 - \Phi(\sqrt{\varepsilon}) \right| \exp(\varepsilon) - \frac{1}{\pi} \sum_{k=0}^{n} \left[ (-1)^{k} \Gamma(k + \frac{1}{2}) / \varepsilon^{k + 1/2} \right] < \Gamma(n + \frac{3}{2}) / \pi \varepsilon^{n+3/2} .
\]

Therefore,
\[
[1 - \Phi(\sqrt{\varepsilon})] \exp(\varepsilon) - \frac{1}{\pi} \sum_{k=0}^{n} \left[ (-1)^{k} \Gamma(k + \frac{1}{2}) / \varepsilon^{k + 1/2} \right] = O(\varepsilon^{-n-1}) , \quad \text{as } \varepsilon \to \infty ,
\]
which permits us to obtain the following asymptotic expansion
\[
1 - \Phi(\sqrt{\varepsilon}) \sim \frac{1}{\pi} \exp(-\varepsilon) \sum_{n=0}^{\infty} \left[ (-1)^{n} \Gamma(n + \frac{1}{2}) / \varepsilon^{n+1/2} \right] , \quad \text{as } \varepsilon \to \infty . \tag{A3}
\]

Inserting (A3) into the right of (A1) gives the following asymptotic expansion of \( \nu(\varepsilon) \)
\[
\nu(\varepsilon) \sim \frac{1}{\pi} \sqrt{\varepsilon / \pi} \sum_{n=0}^{\infty} (-1)^{n} \Gamma(n + \frac{1}{2}) / \varepsilon^{n+1/2} , \quad \text{as } \varepsilon \to \infty . \tag{A4}
\]

It is easy to show that (A4) can be written again in the following form
\[
\nu(\varepsilon) \sim \frac{1}{2} \left\{ 1 - \frac{1}{2 \varepsilon} + \frac{1.3}{(2\varepsilon)^2} - \frac{1.3.5}{(2\varepsilon)^3} + \frac{1.3.5.7}{(2\varepsilon)^4} - \frac{1.3.5.7.9}{(2\varepsilon)^5} + \ldots \right\} , \quad \text{as } \varepsilon \to \infty , \tag{A5}
\]
which shows clearly that \( \nu(\varepsilon) \) is equal to 1/2 when \( \varepsilon \) tends to infinity.

References