Energy in the Dyadosphere of a Reissner-Nordström Black Hole

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It is known that the event horizon of a charged black hole is surrounded by a special region called the dyadosphere, where the electromagnetic field exceeds the Heisenberg-Euler critical value for electron-positron pair production. We obtain the energy distribution in the dyadosphere region of a Reissner-Nordström black hole. We find that the energy-momentum prescriptions of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg give the same acceptable energy distribution.

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I. INTRODUCTION

In a well-known paper, Ruffini [1] introduced a new concept of the dyadosphere of an electromagnetic black hole to explain gamma ray bursts. He defined the dyadosphere as the region just outside the horizon of a charged black hole whose electromagnetic field strength is larger than the well-known Heisenberg-Euler critical value

$$E_{\text{crit}} = \frac{m_e^2 c^3}{e \hbar}$$

for electron-positron pair production ($m_e$ and $e$ respectively denote the mass and charge of an electron). For a Reissner-Nordström space-time, the dyadosphere is described by the radial interval $r_+ \leq r \leq r_{ds}$ where the horizon

$$r_+ = \frac{GM}{c^2} \left( 1 + \sqrt{1 - \frac{Q^2}{GM^2}} \right)$$

forms the inner radius of the dyadosphere, while its outer radius is given by

$$r_{ds} = \sqrt{\left( \frac{h}{m_e c^2} \right) \left( \frac{GM}{c^2} \right) \left( \frac{m_p}{m_e} \right) \left( \frac{e}{q_p} \right) \left( \frac{Q}{\sqrt{GM}} \right)} ,$$

where $M$ and $Q$ are the mass and charge parameters, $m_p = \sqrt{\hbar c/G}$ and $q_p = \sqrt{\hbar c}$ are, respectively, the Planck mass and Planck charge (see in [1]). Ruffini [1], and Preparata et al. [2] have investigated certain properties of the dyadosphere corresponding to the Reissner-Nordström space-time (see also Ruffini et al. [3]). They [3] found that the electron-positron
pair creation process occurs over the entire dyadosphere, excluding the horizon, where the electromagnetic field is Heisenberg-Euler overcritical. The total energy of the electron-positron pairs converted from static electric energy and deposited within the dyadosphere was calculated [3] to be

$$E_{dya} = \frac{1}{2} \frac{Q^2}{r_+} \left(1 - \frac{r_+}{r_{ds}}\right) \left[1 - \left(\frac{r_+}{r_{ds}}\right)^2\right].$$

Several studies show that, in the presence of a strong electromagnetic field, the velocity of light propagation depends on the vacuum polarization states (Birula and Birula [4], Adler [5], De Lorenci et al. [6]). Drummond and Hathrell [7] showed that the effect of vacuum polarization may lead to superluminal photon propagation. Daniels and Shore [8] investigated photon propagation around a charged black hole. Their results show that the effect of the one-loop vacuum polarization on photon propagation in a Reissner-Nordström space-time makes superluminal photon propagation possible. Vacuum polarization effects thus violate the Principle of Equivalence in interacting field theories. It is therefore of interest to examine further properties of the region where the electromagnetic field exceeds the Heisenberg-Euler critical limit. In this paper we investigate the energy distribution in the dyadosphere of a Reissner-Nordström space-time using the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg.

Einstein’s covariant formulation of the energy-momentum conservation law $\nabla_a T^{ab} = 0$ ($T^{ab}$ is the energy-momentum tensor of matter and all nongravitational fields), expressed in the form of the Poynting theorem ($\partial_a \Theta_a^b = 0$, where $\Theta_a^b$ is known as the Einstein energy-momentum complex) to include contributions from the gravitational field, involved the introduction of a pseudotensorial quantity. Owing to the fact that $\Theta_a^b$ is not a true tensor (although it is covariant under linear transformations), Levi-Civita, Schrödinger, and Bauer expressed some doubts about the importance of Einstein’s local energy-momentum conservation laws (see in Cattani and De Maria [9]). Einstein showed that his energy-momentum complex provides a satisfactory expression for the total energy and momentum of isolated systems. This was followed by many more prescriptions: e.g. Landau and Lifshitz, Papapetrou, Weinberg, and many others (for references, see [10]). Most of these prescriptions are coordinate-dependent while others are not. The physical meaning of these was questioned. The large number of energy-momentum prescriptions not only fuelled skepticism that different energy-momentum definitions could give different unacceptable energy distributions for a given space-time, but also lead to diverse viewpoints on the possibility of the localization of energy-momentum. In a series of papers, Cooperstock [11] hypothesized that in general relativity energy and momentum are located only in the regions with nonvanishing energy-momentum tensor and that, consequently, gravitational waves are not carriers of energy and momentum. Although recent results of Xulu [12] and Bringley [13] support this hypothesis, further investigations of this hypothesis are still required.

The main weaknesses of energy-momentum complexes is that most of these restrict one to make calculations in “Cartesian coordinates”. The alternative concept of quasi-local mass is more attractive because it is not restricted to the use of any special coordinate system. There are also a large number of definitions of quasi-local masses. It has been shown
[14] that for a given space-time, many quasi-local mass definitions do not give results which agree. On the other hand, significant contributions of Virbhadra, his co-workers, and some others ([10, 15–17]) encouraged numerous researchers (see [18] and references therein) from many countries to work on this topic. Inspired by Virbhadra’s result, Nester’s research group [19] demonstrated that by associating each of the energy-momentum complexes of Einstein, Landau and Lifshitz, Møller, Papapetrou, and Weinberg with a legitimate Hamiltonian boundary term, each of these complexes may be said to be quasi-local. Quasi-local energy-momentum can be obtained from a Hamiltonian. This important paper [19] dispels the doubts about the physical meaning of these energy-momentum complexes. Hence, below we use four different energy-momentum complexes to obtain the energy distribution in the dyadosphere of a Reissner-Nordstrøm space-time. In the rest of this paper we use $G = 1$, $c = 1$ units, and follow the convention that Latin indices take values from 0 to 3 and Greek indices take values from 1 to 3.

II. THE DE LORENCI ET AL. METRIC

De Lorenci, Figueiredo, Fliche, and Novello [20] calculated the correction for the Reissner-Nordstrøm metric from the first contribution of the Heisenberg-Euler Lagrangian and obtained the following metric:

$$ds^2 = \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6} \right) dt^2 - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

By making $\sigma = 0$ we again obtain the Reissner-Nordstrøm metric. De Lorenci et al. [20] showed that the correction term $\sigma Q^4/(5r^6)$ is of the same order of magnitude as the Reissner-Nordstrøm charge term $Q^2/(2r^2)$.

In order to compute the energy distribution using the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg we are restricted to the use of “Cartesian coordinates”. Therefore we can express the above metric (5) in $T, x, y, z$ coordinates. The coordinate transformation is:

$$T = t + r - \int \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6} \right)^{-1} dr,$$

$$x = r \sin \theta \cos \phi,$$

$$y = r \sin \theta \sin \phi,$$

$$z = r \cos \theta .$$

Thus one has the metric in $T, x, y, z$ coordinates:

$$ds^2 = dT^2 - dx^2 - dy^2 - dz^2.$$
III. ENERGY DISTRIBUTION IN THE REISSNER-NORDSTRÖM DYADOSPHERE

In the following, we use four definitions to compute the energy distribution in the dyadosphere of a Reissner-Nordström black hole.

III-1. Einstein energy-momentum complex

The Einstein energy-momentum complex is given as

$$\Theta^k_i = \frac{1}{16\pi} H^{kl}_{\ i\ \ l},$$

(8)

where

$$H^{kl}_{\ i\ \ l} = - H^{lk}_{\ i\ \ l} = \frac{g_{kn}}{\sqrt{-g}} \left[ -g \left( g^{mn} g^{lm} - g^{ln} g^{km} \right) \right]_{,m}. $$

(9)

$\Theta^0_0$ and $\Theta^0_\alpha$ denote the energy and momentum density components, respectively. (Virbhadra [21] mentioned that although the energy-momentum complex found by Tolman differs in form from the Einstein energy-momentum complex, both are equivalent in import.) The energy-momentum components are expressed by

$$P_i = \int \int \int \Theta^0_i \, dx^1 \, dx^2 \, dx^3. $$

(10)

Further Gauss’s theorem furnishes

$$P_i = \frac{1}{16\pi} \int \int H^0_{\alpha \ i} \mu_\alpha \, dS,$$

(11)

where $\mu_\alpha$ is the outward unit normal vector over the infinitesimal surface element $dS$. $P_\alpha$ give the momentum components $P_1$, $P_2$, $P_3$ and $P_0$ gives the energy.

The only required components of $H^0_{\ i\ \ l}$ in the calculation of the energy are

$$H^0_{01} = \gamma x,$$
$$H^0_{02} = \gamma y,$$
$$H^0_{03} = \gamma z,$$

(12)

where

$$\gamma = \frac{4M}{r^3} - \frac{2Q^2}{r^4} + \frac{2Q^4\sigma}{5r^8}. $$

(13)
For a surface given by the parametric equations $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$ (where $r$ is constant) one has $\mu_\beta = \{x/r, y/r, z/r\}$ and $dS = r^2 \sin \theta d\theta d\phi$. Now using (12) in (11) over a surface $r = \text{const.}$, we obtain

$$E_{\text{Einst}} = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5}.$$  \hspace{1cm} (14)

In the next section we obtain the energy distribution for the same metric in the Landau and Lifshitz formulation.

### III-2. Landau and Lifshitz energy-momentum complex

The symmetric energy-momentum complex of Landau and Lifshitz [22] may be written as

$$L^{ij} = \frac{1}{16\pi} \ell^{ikjl},$$ \hspace{1cm} (15)

where

$$\ell^{ikjl} = -g^{ij}g^{kl} - g^{il}g^{kj}.$$ \hspace{1cm} (16)

$L^{00}$ is the energy density and $L^{0\alpha}$ are the momentum (energy current) density components. $\ell^{mjk}$ has the symmetries of the Riemann curvature tensor. The expression

$$P^i = \int \int \int L^{i0} dx^1 dx^2 dx^3$$ \hspace{1cm} (17)

gives the energy $P^0$ and momentum $P^\alpha$ components. Thus after applying the Gauss theorem, the energy expression is given by

$$E_{\text{LL}} = \frac{1}{16\pi} \int \int \ell^{0\alpha0\beta},\alpha \mu_\beta dS,$$ \hspace{1cm} (18)

where $\mu_\beta$ is the outward unit normal vector over an infinitesimal surface element $dS$. In order to calculate the energy component for the De Lorenci et al. metric expressed by the line element (7), we need the following non-zero components of $\ell^{ikjl}$:

$$\ell^{0101} = -1 + \left( -1 + \frac{x^2}{r^2} \right) \gamma,$$
$$\ell^{0102} = \frac{xy\gamma}{r^2},$$
$$\ell^{0103} = \frac{xz\gamma}{r^2},$$
$$\ell^{0202} = -1 + \left( -1 + \frac{y^2}{r^2} \right) \gamma,$$
$$\ell^{0203} = \frac{yz\gamma}{r^2}.$$ \hspace{1cm} (19)
Equation (15) with Eqs. (16) and (19) gives the energy density component:

\[ L^{00} = \frac{1}{8\pi} \left[ \frac{Q^2}{r^4} - \frac{Q^4\sigma}{r^8} \right]. \] (20)

Using equations (19) in (18) over a surface \( r = \text{const.} \), we obtain

\[ E_{LL} = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5}. \] (21)

Thus we find the same energy distribution we obtained in the last section. In the next section we obtain the energy distribution for the same metric in the Papapetrou formulation.

### III-3. Papapetrou energy-momentum complex

The Papapetrou energy-momentum complex (see in [22]):

\[ \Omega_{ij} = \frac{1}{16\pi} \mathcal{N}^{ijkl},_{kl}, \] (22)

where

\[ \mathcal{N}^{ijkl} = \sqrt{-g} \left( g^{ij} \eta^{kl} - g^{ik} \eta^{jl} + g^{kl} \eta^{ij} - g^{jl} \eta^{ik} \right), \] (23)

is also symmetric in its indices. \( \Omega^{ij} \) are the energy and momentum density components. The energy and momentum components \( P^i \) are given by

\[ P^i = \int \int \int \Omega^0 dx^1 dx^2 dx^3. \] (24)

Applying the Gauss theorem, the energy \( E \) for a stationary metric is given by the expression

\[ E_P = \frac{1}{16\pi} \int \int \mathcal{N}^{00\alpha\beta},_{\beta} \mu_\alpha dS. \] (25)

To find the energy component of the line element (7), we require the following non-zero components of \( \mathcal{N}^{ijkl} \):

\[ \mathcal{N}^{0011} = -1 - \frac{\gamma}{r} + \frac{\gamma x^2}{r^3}, \]
\[ \mathcal{N}^{0012} = \frac{\gamma xy}{r^3}, \]
\[ \mathcal{N}^{0013} = \frac{\gamma xz}{r^3}, \]
\[ \mathcal{N}^{0022} = -1 - \frac{\gamma}{r} + \frac{\gamma y^2}{r^3}, \] (26)
\[ \Omega^0 = \frac{1}{8\pi} \left[ \frac{Q^2}{r^4} - \frac{Q^4\sigma}{r^8} \right]. \]  

(27)

Thus we find the same energy density as we obtained in the last section. We now use Eqs. (26) in (25) over a 2-surface (as in the last section) and obtain

\[ E_{\text{Pap}} = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5}, \]  

which is the same as obtained in the previous section. In the next section we obtain the energy distribution for the same metric in the Weinberg formulation.

**III-4. Weinberg energy-momentum complex**

The symmetric energy-momentum complex of Weinberg (see in [22]) is

\[ W^{ik} = \frac{1}{16\pi} \Delta^{ikl}, \]  

(29)

where

\[ \Delta^{ikl} = \frac{\partial h_a}{\partial x^i} \eta^{lk} - \frac{\partial h_a}{\partial x^l} \eta^{ik} + \frac{\partial h^a}{\partial x^a} \eta^{lk} - \frac{\partial h^a}{\partial x^a} \eta^{ik} + \frac{\partial h^{ik}}{\partial x^l} - \frac{\partial h^{lk}}{\partial x^i} \]  

(30)

and

\[ h_{ij} = g_{ij} - \eta_{ij}. \]  

(31)

\( \eta_{ij} \) is the Minkowski metric. The expression

\[ P^i = \int \int \int W^{0i} dx^1 dx^2 dx^3 \]  

(32)

gives the energy \( P^0 \) and momentum \( P^a \) components. Once more Gauss’s theorem furnishes the following expression for the energy \( E \) of a stationary metric:

\[ E_W = \frac{1}{16\pi} \int \int \Delta^{\alpha k} \mu_\alpha dS. \]  

(33)

To find the energy component of the line element (7), we require the following non-zero components of \( \Delta^{ijk} \):

\[ \Delta^{100} = \gamma x, \]
\[ \triangle^{200} = \gamma y, \]
\[ \triangle^{300} = \gamma z, \]

where \( \gamma \) is given by Eq. (13). To find the energy density component \( W^{00} \), we use Eq. (29) with (30) and (34) and get

\[ W^{00} = \frac{1}{8\pi} \left[ \frac{Q^2}{r^4} - \frac{Q^4\sigma}{r^8} \right], \]

which agrees with energy density components of the Landau-Lifshitz and the Papapetrou energy-momentum complexes. Now using (34) in (33) we obtain

\[ E_W = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5}, \]

which is also the same as in the above sections.

**IV. DISCUSSION**

Misner et al. [23] argued that to look for a local energy-momentum is looking for the right answer to the wrong question. They further argued that energy is only localizable for spherical systems. Cooperstock and Sarracino [24] countered this point of view, arguing that if energy is localizable in spherical systems then it is localizable in any space-times. Bondi [25] pleaded that a non-localizable form of energy is not admissible in general relativity. The viewpoints of Misner et al. discouraged further study of energy localization. On the other hand an alternative concept of energy, the so-called quasi-local energy, was developed. To date, a large number of definitions of quasi-local mass have been proposed. The uses of quasi-local masses to obtain energy in a curved space-time are not limited to a particular coordinate system, whereas many energy-momentum complexes are restricted to the use of “Cartesian coordinates.” Penrose [26] emphasized that quasi-local masses are conceptually very important. Nevertheless, the present quasi-local mass definitions still have inadequacies. For instance, Bergqvist [14] studied seven quasi-local mass definitions and concluded that no two of these definitions agree for the Reissner-Nordstrøm and Kerr space-times. The shortcomings of the seminal quasi-local mass definition of Penrose in handling the Kerr metric are discussed in Bernstein and Tod [27] and in Virbhadra [21]. On the contrary, the remarkable work of Virbhadra, and some others, and recent results of Chang, Nester and Chen have revived interest in various energy-momentum complexes. Recently, Virbhadra stressed that although the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg are nontensorial (under general coordinate transformations), these do not violate the principle of covariance, as the equations describing the conservation laws with these objects are true in any coordinate system.

In this paper we obtained the energy distribution in the De Lorenci et al. space-time using the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and
Weinberg. All four prescriptions give the same distribution of energy \( (E_{\text{Einst}} = E_{LL} = E_{Pap} = E_{W}) \) given as

\[
E = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5}.
\]

(37)

It is obvious that in the dyadospheric region (where \( r \) is small) the last term plays a very important role. As expected, \( \sigma = 0 \) gives the energy distribution for the Reissner-Nordström metric.

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References

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