The Hofstadter Problem for Even Q

M. K. Fung and Y. F. Wang

Department of Physics, National Taiwan Normal University, Taipei, Taiwan 116, R.O.C.

(Received April 21, 2003)

The Hofstadter problem for rational flux with even Q is studied using the chiral representation introduced in a previous paper. We explicitly show that the zero-energy states exist and are degenerate. We also discuss a formal treatment of the perturbations around these zero-energy states.

PACS numbers: 02.30.Ik, 03.65.-w, 03.65.Fd

An electron in a uniform magnetic field is a fundamental problem of classical physics. An exact quantum mechanical solution of the problem was first obtained by Landau [1] with the advent of quantum mechanics. Years later Laughlin [2] solved the problem in the symmetric gauge, and he showed that the explicitly rotationally symmetric solution plays an important role in the quantum Hall effect problem. In 1976 Hofstadter [3] solved the problem numerically where the underlying plane is a square lattice. He obtained a fractal energy spectrum (butterfly pattern), which is fascinating as well as puzzling. A no less startling result was obtained by Wiegmann and Zabrodin [4], who showed that there is a hidden quantum group symmetry for the Hofstadter problem. They showed that, as an alternative, the Bethe ansatz equations can be used to solve the problem for the midband case. We have re-investigated the problem in [5] starting with the formalism of operator algebra. This approach is independent of the choice of gauge for the vector potential. In a chiral representation we do get the same results as [4].

In this article we make a further investigation of the problem using our operator approach. In this problem the uniform magnetic field induces a phase $e^{i\phi}$ in the wavefunction for a path around a plaquette. We can only consider rational flux, that is, $\phi = 2\pi P/Q$, where $P$ and $Q$ are coprime integers. Making use of the periodicity of the lattice, we can reduce the problem to a matrix eigenvalue problem. Periodicity of the lattice introduces a parameter $\lambda$ into the problem. The midband case corresponds to $\lambda = 0$. We continue our study of the problem in the operator approach for the case where $Q$ is even. Indeed, Wen and Zee [7] first studied this case; they proved by a topological method that there exist zero-energy states for the problem. They found a kind of fermionic structure. The main result of our article is to prove by explicit calculation the zero-energy states in this even $Q$ problem.

The Hofstadter problem [3] is the lattice problem of an electron in a uniform magnetic field. The Hamiltonian in this problem can be written as

$$H = T_x + T_y + T_{-x} + T_{-y},$$

(1)
where $T_x$ and $T_y$ are the covariant translation operators along the $x$- and $y$-directions, respectively. The conjugate operators are $T_{-x}$ and $T_{-y}$. We work in units where $e$, $c$, and $\hbar$ are set equal to unity. The covariant translation operators are noncommutative and satisfy the relation

$$T_x T_y = \omega T_y T_x,$$

where $\omega = e^{i\Phi}$. The flux around one plaquette is $\Phi$. We consider the case of rational flux so that $\Phi = 2\pi P/Q$, where $P, Q$ are coprime integers. Generically, $H$ is a $Q \times Q$ matrix. The usual way is to adopt the Landau gauge for the magnetic vector potential. In this way we can represent the two operators $T_x$ and $T_y$ as

$$T_x = e^{ik_x} \begin{pmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix},$$

$$T_y = e^{ik_y} \begin{pmatrix} \omega & 0 & \ldots & 0 \\ 0 & \omega^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$  

Explicitly, they satisfy

$$T_x^Q = e^{iQk_x} \quad \text{and} \quad T_y^Q = e^{iQk_y}.$$  

The eigenvalue problem

$$H \psi = E \psi$$

is calculable as a finite matrix problem. In particular the characteristic equation,

$$\det(H - EI) = 0,$$

will yield the equation for the energy spectrum. We explicitly list the polynomials in $E$ for $Q \leq 4$:

$$E^2 - 4 = \Lambda,$$

$$E^3 - 6E = \Lambda,$$

$$E^4 - 8E^2 + 4 = \Lambda,$$

where $\Lambda = e^{iQk_x} + e^{-iQk_x} + e^{iQk_y} + e^{-iQk_y}$.

It is not possible to work out an explicit general expression for the polynomials in $E$ for any $Q$. The Chambers formula [6] states that the energy polynomial is of $Q$-th order.
in $E$, and the dependence on $\Lambda$ appears only in the coefficient of the constant term. In [5] we give a more cogent proof of the Chambers formula concerning the dependence of $\Lambda$. We also proved that the polynomials are of the form

$$E^Q = \Lambda + 2Q E^{Q-2} + aQ^{-4}E^{Q-4} + \ldots$$

(11)

It is a great innovation that Wiegmann and Zabrodin [4] discovered the inherent quantum group symmetry in this problem. The key point of their study is to identify

$$T_y T_x = \pm q A^2, \quad T_x T_y = \pm q^{-1} D^2,$$

$$T_x + T_y = i(q - q^{-1}) B,$$

$$T_{-x} + T_{-y} = \pm i(q - q^{-1}) C,$$

(12)

where $A, B, C,$ and $D$ are the generators of the algebra $U_q(sl_2)$ (a $q$-deformation of the universal enveloping algebra of $sl_2$) and $q^2 = \omega$. They showed that a cyclic representation of $U_q(sl_2)$ corresponds to the Hofstadter problem in a certain chiral gauge with $\Lambda = 0$.

The introduction of the quantum group symmetry is quite subtle as well as complicated. In [5] we gave an alternative description purely in terms of the operator algebra manipulations. In this situation we worked in a basis defined by

$$T_y T_x |u\rangle = u |u\rangle ,$$

(13)

where $u$ is a phase. We can start with an initial state $|u_0\rangle$. The operator $(T_x + T_y)$ acts as a rotation operator generating the whole set of basis states $|u_0\rangle, |\omega u_0\rangle, |\omega^2 u_0\rangle, \ldots, |\omega^Q u_0\rangle$.

The matrix element of $T_x + T_y$ is quite simple in this basis:

$$\langle \omega^{n+1} u_0 | T_x + T_y | \omega^n u_0 \rangle = e^{i\delta} \left( (\omega^n u_0)^{\frac{1}{2}} + (\omega^n u_0)^{-\frac{1}{2}} \right).$$

(14)

The case is especially simple for $\Lambda = 0$, whence $(T_x + T_y)^Q = T_x^Q + T_y^Q = 0$. We get $\delta = 0$ and $u_0 = -\omega$. In this way we recover the Bethe-like equations in Wiegmann and Zabrodin [4].

The case for even $Q$ is of considerable interest. Wen and Zee [7] showed, by a topological method of calculating the winding number, that for even $Q$ zero energy states exist and are degenerate; around these zero modes Dirac fermions surface in the continuum limit. In this article we employ traditional quantum mechanical methods to derive this important result.

For $Q$ even we have an additional symmetry. It is easily seen that the operator $(T_y T_x)^Q$ anticommutes with $T_x$ and $T_y$ and their conjugates. Thus we can define an operator $\Gamma$ proportional to the operator $(T_y T_x)^Q$ such that

$$\{ \Gamma, T_x \} = \{ \Gamma, T_y \} = \{ \Gamma, H \} = 0 ,$$

(15)

and

$$\Gamma^2 = 1 .$$

(16)
We can go to a basis in which
\[ \Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \]  
where \( I \) is the \( Q/2 \times Q/2 \) identity matrix. We can decompose the operator \( T_x + T_y \) as
\[ T_x + T_y = \begin{pmatrix} 0 & s^\dagger \\ t & 0 \end{pmatrix}, \]  
and
\[ T_{-x} + T_{-y} = \begin{pmatrix} 0 & t^\dagger \\ s & 0 \end{pmatrix}. \]  
The energy hamiltonian becomes
\[ H = \begin{pmatrix} 0 & h^\dagger \\ h & 0 \end{pmatrix}. \]

Now \( T_{-y} T_x \) commutes with \( \Gamma \). This implies that we can simply rearrange the basis as \( |\omega u_0\rangle, |\omega^2 u_0\rangle, \ldots , |\omega^{Q-1} u_0\rangle, |u_0\rangle, |\omega^2 u_0\rangle, \ldots , |\omega^{Q-2} u_0\rangle \), so that \( T_x + T_y \) is of the form of the \( \Gamma \) representation.

The case \( \Lambda = 0 \) is especially simple. The matrix \( s^\dagger \) is then diagonal while the matrix \( t \) has nonzero elements just below the diagonal. That is, the matrix \( h^\dagger \) is triangular
\[ h^\dagger = \begin{pmatrix} u_0^{\frac{1}{2}} + u_0^{-\frac{1}{2}} & qu_0^{\frac{1}{2}} + q^{-1} u_0^{-\frac{1}{2}} & \ldots & 0 & 0 \\ 0 & q^2 u_0^{\frac{1}{2}} + q^{-2} u_0^{-\frac{1}{2}} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & q^{Q/2-1} u_0^{\frac{1}{2}} + q^{-(Q/2-1)} u_0^{-\frac{1}{2}} \end{pmatrix}, \]  
where now \( u_0 = -\omega \); the value of its determinant is then easily calculated as the product of all its diagonal elements. From Eq. (20) it is easy to see that
\[ \det H = (-1)^{Q/2} \det h^\dagger \det h. \]

The determinant of \( h^\dagger \) up to a phase factor is given by
\[ \det h^\dagger \sim (1 + u_0)(1 + \omega^2 u_0) \cdots (1 + \omega^{Q-2} u_0) \sim 2. \]

Thus we have for even \( Q \), and for any \( \Lambda \),
\[ \det H = (-1)^{Q/2} 4 - \Lambda. \]
Zero energy states occur for $\det H = 0$, i.e. $\Lambda = (-1)^{\frac{Q}{2}}4$. It is also evident that the zero energy states are doubly degenerate since the polynomial determining $E$ is of $Q$-th order, and the two bands coalesce at $E = 0$.

For our further discussion we need the formula for $\delta$ and $u_0$ when $\Lambda \neq 0$. Requiring $(T_x + T_y)^Q = T_x^Q + T_y^Q$, which is diagonal, equal to $e^{iQk_x} + e^{iQk_y}$ we can do a short calculation to yield

$$\delta = \frac{k_x + k_y}{2},$$

(25)

and

$$u_0 = -q e^{i/(k_x - k_y)}.$$

(26)

Hence we can write the hamiltonian as

$$H = e^{ik_x}A + e^{-ik_y}A^\dagger + e^{ik_y}A^* + e^{-ik_x}A^T,$$

(27)

where $A$ is of the form

$$A = \begin{pmatrix} 0 & \alpha^\dagger \\ \beta & 0 \end{pmatrix}.$$  

(28)

The $Q/2 \times Q/2$ matrix $\alpha$ is diagonal,

$$\alpha^\dagger = iq^{\frac{1}{2}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{\frac{Q}{2} - 1} \end{pmatrix},$$

(29)

and $\beta$ is of the form

$$\beta = iq^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & \cdots & \omega^{\frac{Q}{2} - 1} \\ 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$  

(30)

We now work in the chiral $\Gamma$-representation so that the eigenvalue problem becomes

$$\begin{pmatrix} 0 & h^\dagger \\ h & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = E \begin{pmatrix} v \\ w \end{pmatrix},$$

(31)

where $v, w$ are $Q/2$ component vectors. Alternatively, we can write this as a system of coupled equations:

$$h^\dagger w = E v,$$

(32)

$$hw = E w.$$  

(33)
The vectors $v$ and $w$ should be equally normalised since $\bar{\varepsilon} h \dagger w = \bar{\omega} hv$, which is real. At the point where $(-1)^{\frac{Q}{2}} 4 - \Lambda = 0$ we have the zero energy solutions. In this case the two equations decouple:

\begin{align}
    h_0 \dagger w_0 &= 0, \\
    h_0 v_0 &= 0. 
\end{align}

(34) (35)

As noted in [7], for the zero energy states we can consider $k_x = k_y = \pi/2$. The matrix $h_0$ can be represented as

\begin{equation}
    h_0 \dagger = i \begin{pmatrix}
        C_0(iq_{\frac{1}{2}}) & -C_1(iq_{\frac{1}{2}}) & 0 & \ldots & 0 \\
        0 & C_2(iq_{\frac{3}{2}}) & -C_3(iq_{\frac{3}{2}}) & \ldots & 0 \\
        \vdots & \vdots & \vdots & \ddots & \vdots \\
        -C_{Q-1}(iq_{\frac{Q}{2}}) & 0 & 0 & \ldots & C_{Q-2}(iq_{\frac{Q}{2}})
    \end{pmatrix},
\end{equation}

(36)

where we define

\begin{equation}
    C_n(x) = q^n x + \frac{1}{q^n x}. 
\end{equation}

(37)

If we denote $w_0^T$ as $(b_1, b_2, \ldots, b_{Q/2})$, then we get

\begin{equation}
    b_{i+1}/b_i = C_{2i-2}(iq_{\frac{i}{2}})/C_{2i-1}(iq_{\frac{i}{2}}),
\end{equation}

(38)

and $v_0^T = (a_1, a_2, \ldots, a_{Q/2})$ satisfying the relation

\begin{equation}
    a_i/a_{i+1} = C_{2i-1}(iq_{\frac{i}{2}})/C_{2i}(iq_{\frac{i}{2}}). 
\end{equation}

(39)

Just as before, $v_0$ and $w_0$ should be normalised equally. For convenience we take them both to be normalised to unity.

In the vicinity of the zero energy states we take $k_x = \pi/2 + \kappa$ and $k_y = \pi/2$ and do a perturbative calculation. We expect the resulting $E^2$ to be proportional to $\kappa^2$ because of the inherent rotational symmetry. It is difficult to do a perturbation calculation for $H$ because of the degenerate zero energy states. However, we can work with the reduced hamiltonian

\begin{align}
    (h_0 \dagger + h_0 \dagger)(w_0 + \delta w) &= E(v_0 + \delta v), \\
    (h_0 + h_0')(v_0 + \delta v) &= E(w_0 + \delta w). 
\end{align}

(40) (41)

With $v_0$ and $w_0$ normalised to unity we get

\begin{equation}
    E = \bar{\omega} h \dagger v_0. 
\end{equation}

(42)

There is a relative phase ambiguity between $v_0$ and $w_0$. This is resolved by requiring $E$ to be real. Thus we get an energy $E$ linearly dependent on $\kappa$. The coefficient of proportionality is dependent on $Q$ and does not have a simple expression.
The discussion of the perturbative treatment here is at most formal, since we do not have a compact formula for the normalised zero-energy wavefunctions. This may be useful for numerical calculations in which normalised wavefunctions can be easily obtained. However, in this article we do show explicitly that we can get the zero-energy states by analytical calculation, in contrast to the topological method of [7]. This article makes a small contribution to our further understanding of the Hofstadter problem in some particular cases. It also demonstrates that the algebraic formulation can be viable; it gives another useful point of view for discussing the Hofstadter problem.

Acknowledgments

This work was supported in part by the National Science Council of the R.O.C. under the contract NSC-90-2112-M-003-011.

References