Coherent Soliton Structures with Chaotic and Fractal Behaviors in a Generalized (2+1)-Dimensional Korteweg de-Vries System

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In higher dimensions there are abundant coherent soliton excitations. In this work, we reveal a novel phenomenon whereby the localized excitations show chaotic and fractal behaviors in some (2+1)-dimensional physical models. To clarify this interesting phenomenon, we take the generalized (2+1)-dimensional Korteweg de-Vries systems:

\[ v_t + av_{xxx} + bv_{yyy} + cv_x + dv_y = 3a(uv)_x + 3b(vw)_y, \]

\[ v_x = u_y, \quad v_y = w_x \]

as a concrete example. By means of a new variable separation approach, a quite general variable separation solution of this system is derived. Along with the usual localized coherent soliton excitations such as dromions, lumps, rings, peakons, and oscillating soliton excitations, some new excitations like chaos and fractals are derived by introducing some types of lower dimensional chaotic and fractal patterns.

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I. INTRODUCTION

Three important aspects of nonlinear science are solitons, chaos, and fractals [1]. They are widely applied in many natural sciences such as chemistry, biology, mathematics, communications, and especially in almost all branches of physics such as fluid dynamics, plasma physics, field theory, optics, and condensed matter physics [2]. Conventionally, the three aspects are treated independently, since one often considers that solitons are the basic excitations in an integrable model while chaos and fractals are elementary behaviors of a non-integrable system. In other words, one does not analyze the possibility of the existence of chaos and fractals in a soliton system. However, the above consideration may not be complete, especially in some higher dimensions. In our recent study of soliton systems, we have found that some characteristic lower dimensional arbitrary functions exist in the exact excitation of certain two dimensional integrable models. This means that any lower dimensional chaotic and/or fractal solution can be used to construct an exact solution of
a higher dimensional integrable model, which also implies that any exotic behavior may propagate along this characteristic.

To verify the above viewpoint, we take the generalized (2+1)-dimensional Korteweg de-Vries system (GKdV) as a concrete example [3]:

\[ v_t + av_{xxx} + b v_{yyy} + c v_x + d v_y - 3a(uv)_x - 3b(vw)_y = 0, \]  
\[ v_x = u_y, \quad v_y = w_x, \]

where \(a, b, c\) and \(d\) are arbitrary constants. When \(c = d = 0\), the (2+1)-dimensional GKdV system degenerates to the usual two dimensional Korteweg de-Vries (2DKdV) equation [4]:

\[ v_t + av_{xxx} + b v_{yyy} - 3a(uv)_x - 3b(vw)_y = 0, \]  
\[ v_x = u_y, \quad v_y = w_x, \]

which is an isotropic Lax extension of the well known (1+1)-dimensional KdV equation. Many types of soliton solutions of the 2DKdV equation have now been studied by many authors. For instance, Boiti, Leon, Manna and Pempinelli [3] solved the 2DKdV equation via the inverse scattering transformation. Tagami [5], and Hu and Li [6] obtained its soliton-like solution by means of the Bäcklund transformation. Hu [7] has also given a nonlinear superposition formula for the 2DKdV equation. Ohta [8] obtained the Pfaffian solutions for this mode. Radha and Lakshmanan [9] constructed its dromion solution from its bilinear form after analyzing its integrability aspects. Lou [10] obtained some new special types of multi-soliton solutions of this equation, by using the standard truncated Painlevé analysis, and localized excitations by means of a variable separation approach.

However, the (2+1)-dimensional GKdV system has not been studied so well, although Boiti, Leon, Manna and Pempinelli [3] solve this system via the inverse scattering transformation, and Radha and Lakshmanan have constructed its multidromion solutions [9] while Zhang [11] obtained an exact solution for this system based on an extended homogeneous balance approach. In this paper, we investigate the GKdV system further. More attention is paid to its localized coherent structures, especially the localized coherent soliton structure with some novel properties like fractal and chaotic properties. In Sec.2, we outline the main procedures of the variable separation approach and apply this approach to the GKdV system. In Sec.3, we discuss the abundant localized solution structures based on results obtained by the variable separation approach. A brief discussion and a summary are given in the last section.

II. VARIABLE SEPARATION APPROACH AND ITS APPLICATION TO THE (2+1)-DIMENSIONAL GKDV SYSTEM

II-1. General theory of the variable separation approach

It is well known that solving nonlinear physical models is much more difficult than solving linear ones. In linear physics, the Fourier transformation (FT) and the variable
separation approach (VAS) are the two most important solution methods. The celebrated inverse scattering transformation (IST) can be viewed as an extension of the Fourier transformation method. However, it is difficult to extend the VAS to nonlinear physics. Recently, two kinds of “variable separating” procedures have been established. The first method is called symmetry constraints or nonlinearization of the Lax pairs [12], because this type of procedure is used only for integrable models which possess Lax pairs. Lou and Chen extended the method to some nonintegrable models. This method can be equivalently called the “formal variable separation approach” (FVSA) [13]. The independent variables of a reduced field in the FVSA have not been totally separated; the reduced field satisfies some lower-dimensional equations. The second kind of variable separation method was first established for the (2+1)-dimensional Davey-Stewartson (DS) equations and the asymmetric DS equation [14], and then recently revisited and developed for various (2+1)-dimensional models, like the generalized Ablowitz-Kaup-Newell-Segur (AKNS) system, the nonlinear Schrödinger equation (NLS), the usual and asymmetric Nizhnik-Novikov-Vesselov (NNV) system, and the long dispersive wave system and Maccari system [15-21]. The main idea is that by solving the bilinear equations or higher multi-linear equations of the original models and introducing a prior ansatz, some special types of exact solutions of (2+1)-dimensional nonlinear models can be obtained from some (1+1)-dimensional variable separation fields. Here we briefly describe the basic procedures of the variable separation approach. For a general nonlinear physical system,

\[ P(v) \equiv P(x_0 = t, x_1, x_2, \ldots, x_n, v, v_{x_1}, v_{x_2}, \ldots), \]

(5)

where \( v = [v_1, v_2, \ldots, v_q]^T \) (∧ indicates the transposition of a matrix), \( P(v) = (P_1(v), P_2(v), \ldots, P_q(v))^T \), and \( P_i(v) \) are polynomials of \( v_i \) and their derivatives.

First, we make a Backlund transformation

\[ v_i = \sum_{j=0}^{\alpha_i} v_{ij} f^{j-\alpha_i}, \quad i = 1, 2, \ldots, q, \]

(6)

where the \( v_{i\alpha} \) are arbitrary known seed solutions of Eq. (5). In the usual cases, \( \alpha_i \) should be as small as possible, since substituting Eq. (6) into Eq. (5) can yield relatively simple multi-linear equations in this situation and is determined by the leading term analysis (suppose \( f \sim 0 \)). Inserting Eq. (6) with \( \alpha_i \) into Eq. (5) and requiring the leading and sub-leading terms to vanish, we can then derive \( \{v_{ij}, j = 0, 1, 2, \ldots, \alpha_i - 1\} \). If the original model is integrable, this procedure will result in its bilinear or higher multi-linear equations.

Second, after obtaining multi-linear equations for the original system, we select an appropriate variable separation hypothesis. For integrable models, it can be chosen as modifying Horita’s multi-soliton forms. For instance for some celebrated physical models [22-24] we often take for \( f \) an ansatz such as

\[ f = a_0 + a_1 p(x, t) + a_2 q(y, t) + a_3 p(x, t) q(y, t), \]

(7)

where the variable separation functions \( p(x, t) \equiv p \) and \( q(y, t) \equiv q \) are only functions of \( (x, t) \) and \( (y, t) \), respectively, and \( a_0, a_1, a_2, \) and \( a_3 \) are arbitrary constants. When \( p \) and \( q \) are set as exponential functions, Eq. (7) is just Hirota’s two-soliton form.
Finally, we determine the variable separation equations, which the variable separation functions \( p \) and \( q \) should satisfy after substituting the ansatz (7) into the multi-linear equations. It is worth mentioning that this procedure is tedious, since different physical models must be disposed in different ways. In the next subsection, we apply this approach to a \((2+1)\)-dimensional GKdV system. The variable separation approach shows its efficiency in searching for localized excitations of nonlinear physical models.

II-2. Variable separation approach for the \((2+1)\)-dimensional GKdV system

According to the above procedures, we first take the following Backlund transformation of \( v, u, \) and \( w \) in Eqs. (1) and (2)

\[
v = \sum_{j=0}^{\alpha_1} v_j f_j^{\alpha_1}, \quad u = \sum_{j=0}^{\alpha_2} u_j f_j^{\alpha_2}, \quad w = \sum_{j=0}^{\alpha_3} w_j f_j^{\alpha_3},
\]

where \( v_0, u_0, \) and \( w_0 \) are arbitrary seed solutions of the GKdV system. By using the leading term analysis, we obtain

\[
\alpha_1 = \alpha_2 = \alpha_3 = 2.
\]

Substituting Eqs. (8) and (9) directly into the Eqs. (1) and (2), and considering the fact that the functions \( v_2, u_2 \) and \( w_2 \) are seed solutions of the model, yields

\[
\sum_{i=0}^{4} P_{1i} f_i^{4-5} = 0, \quad \sum_{i=0}^{2} P_{2i} f_i^{4-3} = 0, \quad \sum_{i=0}^{2} P_{3i} f_i^{4-3} = 0,
\]

where \( P_{1i}, P_{2i}, P_{3i} \) are functions of \( \{v_j, u_j, w_j f, j = 0, 1\} \) and their derivatives. Due to the complexity of the expressions for \( P_{1i}, P_{2i}, \) and \( P_{3i} \), we omit their concrete forms. By requiring the leading and sub-leading terms of Eqs. (10) and (11) to vanish, the functions \( \{v_j, u_j, w_j f, j = 0, 1\} \) are determined. Inserting all the results into Eq. (8) and rewriting its form, the Backlund transformation becomes

\[
v = -2(\ln f)_{xy} + v_2, \quad u = -2(\ln f)_{xx} + u_2, \quad w = -2(\ln f)_{yy} + w_2.
\]

For convenience of discussion, we choose the seed solutions \( v_2, u_2 \) and \( w_2 \) as

\[
v_2 = 0, \quad u_2 = p_0(x, t), \quad w_2 = q_0(y, t),
\]

where \( u_2(x, t) \) and \( w_2(y, t) \) are arbitrary functions of the indicated arguments.

Substituting Eqs. (12) and (13) into Eqs. (1) and (2), Eq.(2) is now an identical equation, while Eq. (1) assumes the following symmetric form

\[
(c - 3ap_0)(2f f_x f_{xy} - f_x^2 f_{xy} - 2f_x f_y + f f_{xx} f_y) + (d - 3bq_0)(2f y f_y f_{xy} - f^2 f_{yyx} - 2f_y^2 f_x + f f_{yy} f_x)
\]
COHERENT SOLITON STRUCTURES WITH

which can be rewritten as a multi-linear equation with respect to

Substituting the ansatz (7) into Eq. (14) yields

According to Eq. (16), we simply take the following variable separated equations:

First, we set the coefficient of the \( f^2 \) to zero, namely

Substituting the ansatz (7) into Eq. (15) yields

According to Eq. (16), we simply take the following variable separated equations:

Second, inserting the ansatz (7) with Eqs. (17) and (18) into Eq. (14), we find that Eq. (14) is satisfied identically. Also it is not easy to obtain general solutions to Eqs. (17) and (18) for any fixed \( p_0 \) and \( q_0 \). However, we can treat the problem in an alternate way. Because \( p_0 \) and \( q_0 \) are arbitrary seed solutions, we can view \( p \) and \( q \) as arbitrary functions of \( \{x, t\} \) and \( \{y, t\} \), respectively, then the seed solutions \( p_0 \) and \( q_0 \) can be fixed by Eqs. (17) and (18).

Finally, we obtain a quite general excitation of the GKdV system

with two arbitrary functions \( p(x, t) \), \( q(y, t) \), and \( a_0, a_1, a_2, a_3 \) being arbitrary constants.
III. SOME COHERENT SOLITON STRUCTURES WITH CHAOTIC AND FRACTAL BEHAVIORS IN THE (2+1)-DIMENSIONAL GKDV SYSTEM

It is interesting that expression (19) is valid for many (2+1)-dimensional models such as the DS equation, the NNV system, the ANNV equation, the ADS model, the GAKNS, the dispersive long wave equation, etc. [15, 16, 19, 21]. Because of the arbitrariness of the functions \( p \) and \( q \) included in Eq. (19), the quantity \( v \) possesses quite rich structures. For instance, as mentioned in Refs. [15, 16, 19, 21], when the functions \( p \) and \( q \) are selected appropriately, we can obtain many kinds of localized solutions, like multi-soliton solutions, multi-dromion and dromion lattice solutions, multiple ring soliton solutions, multiple peakon solutions, and so on. Since these types of localized solutions have been discussed widely for other models, we omit all the stable localized coherent soliton structures here.

An important matter is whether we can find some new types of solitons which possess chaotic and/or fractal behavior, say, the chaotic and/or fractal localized excitations for the soliton system. The answer is apparently positive, since \( p(x,t) \) and \( q(y,t) \) are arbitrary functions. There are various chaotic and fractal dromion and lump excitations, because any types of (1+1)- and/or (0+1)-dimensional chaos and fractal models can be used to construct localized excitations of higher dimensional models. Some interesting possible chaotic and fractal patterns are cited here. For simplification in the following discussion, we set \( a_0 = 0 \), \( a_1 = a_2 = 1 \), and \( a_3 = 2 \) in the expression (19).

III-1. Chaos

(1) Chaotic dromions
In (2+1)-dimensions, one of the most important nonlinear solutions is the dromion excitation, which is localized in all directions. Now we set \( p \) and \( q \) to be

\[
p = \exp(x), \quad q = 1 + (100 + f(t)) \exp(y),
\]

where \( f(t) \) is an arbitrary function of time \( t \). From the excitation (19) with equation (22) one knows that the amplitude of the dromion is determined by the function \( f(t) \). If we select the function \( f(t) \) as a solution of a chaotic system, then we can obtain a type of chaotic dromion solution. In Fig. 1a, we plot the shape of the dromion for the physical quantity \( v \) shown by expression (19) at a fixed time (for \( f(t) = 0 \)) with condition (22). The amplitude \( A \) of the dromion related to \( \parallel a \parallel \) is changed chaotically with \( f(t) \), as depicted in Fig. 1b, where \( f(t) \) is a solution of the following Lorenz system [25]

\[
\begin{align*}
f_t &= -10(f - g), \quad g_t = f(60 - h) - g, \quad h_t = fg - \frac{8}{3}h.
\end{align*}
\]

This fact hints that the functions \( p \) or \( q \) may take a more general form. For example, if we set \( p \) and \( q \) to be \([f_3(t) > 0, f_7(t) > 0]\)

\[
p = \frac{f_1(t)}{f_2(t) + \exp(f_3(t)(x + f_4(t)))}, \quad q = \frac{f_5(t)}{f_6(t) + \exp(f_7(t)(x + f_8(t)))},
\]

(24)
with $f_i(t), i = 1, 2, \ldots, 8$ being chaotic solutions, then solution (19) becomes a chaotic dromion which may be chaotic in different ways. The amplitude of the dromion (19) with Eq. (24) will be chaotic if $f_1(t), f_2(t), f_5(t)$ and/or $f_6(t)$ are chaotic, while $f_4(t)$ and/or $f_8(t)$ are chaotic if the position of the dromion location becomes chaotic. The shape (width) of the dromion may be chaotic if the functions $f_3(t)$ and/or $f_7(t)$ are chaotic. Since a detailed physical discussion has been given in Ref. [26], we omit the related plots here.

(2) Chaotic line solitons

It is interesting that the localized excitations are not only chaotic with time $t$, but also with space, say, in the direction $x$ and/or $y$. If either one of $p$ and $q$ is selected as a localized function while other one is a chaotic solution of some (1+1)-dimensional (or (0+1)-dimensional) nonintegrable models, then the excitation (19) becomes a chaotic line soliton which may be chaotic in the $x$ or $y$ direction.

For example, we take $p$ or $q$ as the solution of $(\zeta = x + \omega t, \eta = y + \nu t)$

$$p_{\zeta\zeta\zeta} = \frac{p_{\zeta} p_{\zeta} + (c + 1) p_{\zeta}^2}{p} - (p^2 + b(c + 1)) p_{\zeta} - (b + c + 1) p_{\zeta\zeta} + (b(a - 1) - p^2) c p,$$  

(25)

$$q_{\eta\eta} = \frac{q_{\eta} q_{\eta} + (\gamma + 1) q_{\eta}^2}{q} - (q^2 + \beta(\gamma + 1)) q_{\eta} - (\beta + \gamma + 1) q_{\eta\eta} + (\beta(a - 1) - q^2) \gamma q,$$  

(26)

where $a, b, c, \alpha, \beta, \gamma, \omega,$ and $\nu$ are all arbitrary constants. Actually, Eq. (25) or Eq. (26) is also equivalent to the celebrated Lorenz system

$$p_{\zeta} = -c(p - g), \quad g_{\zeta} = p(a - h) - g, \quad h_{\zeta} = pg - bh,$$  

(27)

upon eliminating the functions $g$ and $h$ in Eq.(27). Fig. 2(a) is a plot of the chaotic line soliton.
FIG. 2: (a) A plot of the chaotic line soliton structure for the physical quantity $v$ given by expression (19) with the conditions (27), (28), and (29). (b) A typical plot of the chaotic solution $p$ in the Lorenz system (27).

The soliton solution expressed by Eq. (19) with the selections: $p$ is a chaotic solution of the Lorenz system (27) and

$$q = \tanh(y).$$

while the parameters are taken as

$$a = \alpha = 60, \quad b = \beta = \frac{8}{3}, \quad c = \gamma = 1, \quad t = 0.$$  \hfill (29)

A typical plot of the chaotic solution $p$ in the above Lorenz system is presented in Fig. 2(b).

(3) Non-localized chaotic patterns

Certainly, if both $p$ and $q$ are selected as chaotic solutions of some lower dimensional nonintegrable models, then the excitation (19) becomes chaotic in both the $x$ and $y$ directions, which yields non-localized chaotic-chaotic patterns. Figure 3(a) shows a plot of a special chaotic-chaotic pattern expressed by Eq. (19), $p$ and $q$ being given by Eqs. (25) and (26), with the selections Eq. (29) and

$$a_0 = 200, \quad a_1 = a_2 = 1, \quad a_3 = 0.$$  \hfill (30)

Figure 3(b) shows the corresponding plot for two typical chaotic solutions of the Lorenz model (27) with Eq. (29).

If only one of $p$ and $q$ is selected as a chaotic solution for some lower dimensional nonintegrable models, while the other is some periodic function, then the excitation (19) becomes chaotic in one direction and periodic in the other direction, which yields a non-localized chaotic-periodic pattern. Figure 4(a) shows a plot of the special chaotic–periodic pattern expressed by Eq. (19), with Eq. (30) and $p$ being the chaotic solution of Eq. (25), also $a = 60, b = 8/3, c = 10$, and $q$ is the periodic solution of Eq. (26) with

$$\alpha = 350, \quad \beta = 8/3, \quad \gamma = 10.$$  \hfill (31)
FIG. 3: (a) A plot of the special chaotic–chaotic pattern expressed by Eq. (19), with \( p \) and \( q \) being chaotic solutions to Eqs. (25) and (26) under the selections Eqs. (29) and (30). (b) A plot for a typical chaotic solution of the Lorenz system (27) with Eq. (29).

FIG. 4: (a) A plot of the special chaotic–periodic pattern expressed by Eq. (19), with Eq. (30) and \( p \) being the chaotic solution of Eq. (25), with \( a = 60, b = 8/3, c = 10 \), while \( q \) is the periodic solution of Eq. (26) with condition Eq. (31). (b) A plot of a typical periodic Lorenz system solution (27) with Eq. (31).

Figure 4(b) is a plot of two typical periodic solutions of the Lorenz model (27) with Eq. (31).

III-2. Fractals

(1) Regular fractal lumps and dromions
Now we discuss the localized coherent excitation with fractal properties. The lump solution (algebraically localized in all directions) is one type of significant localized excitation. In Refs. [26–28], the authors found that many lower-dimensional piecewise smooth functions with a fractal structure can be used to construct exact solutions of a higher-dimensional soliton system, which also possesses a fractal structure. This situation also occurs in a
FIG. 5: (a) A fractal lump structure for the potential $v$ with the conditions (32)-(33) at $t = 0$. (b) A density plot of the fractal lump related to (a) in the region $\{x = [-0.45, 0.45], y = [-0.45, 0.45]\}$.

$(2+1)$-dimensional GKdV system. If the functions $p$ and $q$ are selected appropriately, we can find some types of lump solutions with fractal behaviors. Figure 3(a) shows a fractal lump structure for the physical quantity $v$ in Eq. (19), where $p$ and $q$ in expression (19) are simply selected as

$$p = 1 + \frac{|x - c_1t|}{1 + (x - c_1t)^4}(\cos(\ln(x - c_1t)^2))^2,$$

$$q = 1 + \frac{|y - c_2t|}{1 + (y - c_2t)^4}(\cos(\ln(y - c_2t)^2))^2,$$

at $t = 0$. In figure 5(a), we can see that the solution is localized in all directions. Near the center there are infinitely many peaks which are distributed in a fractal manner. In order to investigate the fractal structure of the lump, we should look at the structure more carefully. Figure 5(b) presents a density plot of the structure of the fractal lump in the region $\{x = [-0.045, 0.045], y = [-0.045, 0.045]\}$. More detailed studies will show us the interesting self-similar structure of the lump. For example, if we reduce the region in figure 5(b) to $\{x = [-0.002, 0.002], y = [-0.002, 0.002]\}, \{x = [-0.000085, 0.000085], y = [-0.000085, 0.000085]\}, \{x = [-3.6 \times 10^{-6}, 3.6 \times 10^{-6}], y = [-3.6 \times 10^{-6}, 3.6 \times 10^{-6}]\}$ and so on, we will find structures totally similar to those plotted in figure 5(b).

The dromion solution (exponentially localized in all directions) is another type of significant localized excitation. If the arbitrary functions $p$ and $q$ are appropriately selected, we find that some special types of fractal dromions for the physical quantity $v$ (19) are revealed. For example, if we take

$$p = 1 + \exp[\sqrt{(x - c_1t)^2(1 + \sin(\ln((x - c_1t)^2)))}],$$

$$q = 1 + \exp[\sqrt{(y - c_2t)^2(1 + \sin(\ln((y - c_2t)^2)))}],$$

The dromion solution (exponentially localized in all directions) is another type of significant localized excitation. If the arbitrary functions $p$ and $q$ are appropriately selected, we find that some special types of fractal dromions for the physical quantity $v$ (19) are revealed. For example, if we take
FIG. 6: (a) A plot of a fractal dromion structure for the potential \( v \) given by the solution (19), with the conditions (34)-(35), at \( t = 0 \). (b) A density plot of the fractal structure of the dromion in the region \( \{ x = [-0.12, 0.12], y = [-0.12, 0.12] \} \).

then we can obtain a simple fractal dromion. Figure 6(a) shows a plot of this special type of fractal dromion structure for the potential \( v \) given by Eq. (19), with the conditions in Eqs. (34)-(35) at time \( t = 0 \). Figure 6(b) is a density plot of the fractal structure of the dromion in the region \( \{ x = [-0.12, 0.12], y = [-0.12, 0.12] \} \). To observe the self-similar structure of the fractal dromion more clearly, one may enlarge a small region near the center of figure 6(b). For instance, if we reduce the region of figure 4(b) to \( \{ x = [-0.002, 0.002], y = [-0.002, 0.002] \} \), \( \{ x = [-2.5 \times 10^{-14}, 2.5 \times 10^{-14}], y = [-2.5 \times 10^{-14}, 2.5 \times 10^{-14}] \} \) and so on, we find structure totally similar to that presented in figure 6(b).

(2) Stochastic fractal dromions and lumps

In addition to self-similar regular fractal dromions and lumps, the lower-dimensional stochastic fractal functions may also be used to construct high-dimensional stochastic fractal dromion and lump excitations. For instance, one of the best known stochastic fractal functions is the Weierstrass function:

\[
G(X) = \sum_{k=0}^{L} (3/2)^{-k/2} \sin((3/2)^k X), \quad N \to \infty,
\]

where the independent variable \( X \) may be a suitable function of \( \{ x + at \} \) and/or \( \{ y + bt \} \), say \( X = x + at \) and \( X = y + bt \), as in the functions \( p \) and \( q \), respectively, for the following selection (37). If the Weierstrass function is included in the dromion or lump excitations, then we can derive stochastic fractal dromions and lumps. Figure (7) shows a plot of a typical stochastic fractal lump solution, determined by Eq. (19), with the conditions Eq. (36) and

\[
p = G(x + at) + (x + at)^2 + 10^2, \quad q = G(y + bt) + (y + bt)^2 + 10^2,
\]

(37)
FIG. 7: A plot of a typical stochastic fractal lump solution determined by Eq.(19) with the selections (36) and (37).

at $t = 0$. In figure (7), the vertical axis denotes the quantity $V$, which is only a re-scaling of the potential $v$: $v = V \times 10^{-7}$.

IV. SUMMARY AND DISCUSSION

In summary, with help of the variable separation approach, the (2+1)-dimensional GdV system is solved. Abundant localized coherent soliton structures of the solution (19), such as multi-dromion, multi-ring, multi-lump solutions, peakons, breathers, instantons, etc., can be easily constructed by selecting appropriate arbitrary functions.

In addition to the usual localized coherent soliton structures, we find some new localized excitations—the chaotic and fractal soliton solutions for the (2+1)-dimensional GdV system. As is known, chaos and fractals not only belong to the realms of mathematics and computer graphics, but also exist nearly everywhere in nature, such as in fluid turbulence, crystal growth patterns, human veins, fern shapes, galaxy clustering, cloud structures and in numerous other examples. Conventionally, chaos and fractals are opposite to solitons in nonlinear science, since solitons are the representatives of an integrable system while chaos and fractals represent non-integrable systems. However, in this paper, we find some chaotic and fractal structures for localized solutions of the (2+1)-dimensional integrable GdV model. Why do localized excitations possess some kinds of chaotic and/or fractal behaviors? If one considers the boundary and/or initial conditions of the chaotic and/or fractal solutions obtained here, one can find that the initial and/or boundary conditions possess chaotic and/or fractal properties. In other words, the chaotic and/or fractal behaviors of the localized excitations of the integrable models come from the nonintegrable boundary and/or initial conditions.

Because of the numerous applications of the soliton, chaos, and fractal theories in many physics fields, such as fluid dynamics, plasma physics, field theory, optics, etc., it is important to learn more about the localized chaotic and/or fractal excitations of an
“integrable” physical model with “nonintegrable” boundary and/or initial conditions, such as chaotic and/or fractal boundaries and/or initial conditions.

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