Lame Function and Perturbed Solutions to the NLS Equation and the Coupled NLS Equations

Zuntao Fu,* Shikuo Liu, and Shida Liu

School of Physics, Peking University, Beijing, 100871, China
(Received February 17, 2003)

In this paper, on the basis of the Lame equation and Lame functions, the Jacobi elliptic function expansion method and the perturbation method are applied to the nonlinear Schrödinger (NLS) equation and the coupled nonlinear Schrödinger equations to derive their perturbed solutions.

PACS numbers: 02.90.+p

I. INTRODUCTION

The finding of exact solutions to nonlinear evolution equations in nonlinear studies is worthy of more effort, because there are so many coherent structures and patterns. Many methods, such as the homogeneous balance method [1-3], the hyperbolic tangent function expansion method [4-6], the nonlinear transformation method [7,8], the trial function method [9,10], the sine-cosine method [11], the Jacobi elliptic function expansion method [12,13], and so on [14-17], have been proposed and applied to get many exact solutions, from which one can see the richness of the structures that exists in the different nonlinear wave equations. In order to discuss the stability of these solutions, it is necessary to superimpose a small disturbance, and then analyze the evolution of the small disturbance [18,19]. This is equivalent to expanding the solutions of the nonlinear evolution equations as a power series in terms of a small parameter ε, so that multi-order exact solutions (including perturbed solutions) can be derived. In this paper, on the basis of the Jacobi elliptic function expansion method, the multi-order exact solutions of some nonlinear evolution equations are obtained by means of the Jacobi elliptic functions and the Lame function [19,20].

II. LAME EQUATION AND LAME FUNCTIONS

Usually, the Lame equation [20] in terms of y(x) is written as

\[ \frac{d^2y}{dx^2} + [\lambda - n(n + 1)m^2 \text{sn}^2 x]y = 0, \]  

(1)

where λ is an eigenvalue, n is a positive integer, and snx is the Jacobi elliptic sine function with modulus m (0 < m < 1).
Set
\[ \eta = \text{sn}^2 x, \quad (2) \]
then the Lame equation (1) becomes
\[ \frac{d^2 y}{d\eta^2} + \frac{1}{2} \left( \frac{1}{\eta} + \frac{1}{\eta - 1} + \frac{1}{\eta - h} \right) \frac{d y}{d\eta} - \frac{\mu + n(n + 1)\eta}{4\eta(\eta - 1)(\eta - h)} y = 0, \quad (3) \]
where
\[ h = \frac{m^2}{2} > 1, \quad \mu = -h\lambda. \quad (4) \]
Equation (3) is a kind of Fuchs-typed equation with four regular singular points \( \eta = 0, 1, h, \) and \( \eta = \infty; \) a solution to the Lame equation (3) is known as a Lame function.

For example, when \( n = 2, \lambda = 1 + m^2, \) i.e. \( \mu = -(1 + m^{-2}), \) the Lame function is
\[ L_2^s(x) = (1 - \eta)^{1/2}(1 - h^{-1}\eta)^{1/2} = \text{cn}\text{dn} x \quad (5) \]
and its corresponding equation is
\[ \frac{d^2 L_2^s}{dx^2} + [(1 + m^2) - 6m^2\text{sn}^2 x] L_2^s = 0. \quad (6) \]
Similarly, when \( n = 2, \lambda = 1 + 4m^2, \) the Lame function is
\[ L_2^c(x) = \text{sn}\text{dn} x, \quad (7) \]
and its corresponding equation is
\[ \frac{d^2 L_2^c}{dx^2} + [(1 + 4m^2) - 6m^2\text{sn}^2 x] L_2^c = 0, \quad (8) \]
and when \( n = 2, \lambda = 4 + m^2, \) the Lame function is
\[ L_2^d(x) = \text{sn}\text{cn} x, \quad (9) \]
and its corresponding equation is
\[ \frac{d^2 L_2^d}{dx^2} + [(4 + m^2) - 6m^2\text{sn}^2 x] L_2^d = 0. \quad (10) \]
There is still another case where \( n = 3, \lambda = 4(1 + m^2), \) i.e. \( \mu = -4(1 + m^{-2}). \) The Lame function is
\[ L_3(x) = \eta^{1/2}(1 - \eta)^{1/2}(1 - h^{-1}\eta)^{1/2} = \text{sn}\text{cn}\text{dn} x. \quad (11) \]
In the above, \( \text{cn} \) and \( \text{dn} \) are the Jacobi elliptic cosine function and the Jacobi elliptic function of the third kind [19,20], respectively. In the next sections, we will apply the Lame functions \( L_2^s(x), L_2^c(x) \) and \( L_2^d(x) \) and their corresponding Lame equations to solve the NLS equation and the coupled NLS equations and to derive their corresponding perturbed solutions. The application of the Lame function \( L_3(x) \) will be illustrated elsewhere.
III. APPLICATION TO THE NLS EQUATION

In this section, we consider the application of the Lame equations to the NLS equation

\[ i\frac{\partial u}{\partial t} + \alpha \frac{\partial^2 u}{\partial x^2} + \beta |u|^2 u = 0. \]  

(12)

We seek travelling wave solutions of the following form

\[ u = \phi(\xi)e^{i(kx-\omega t)}, \quad \xi = p(x-ctgt), \]  

(13)

where \( k \) and \( c_g \) are the wave number and wave group speed, \( \omega \) is the angular frequency, and \( \phi(\xi) \) is a real function.

Substituting (13) into (12), we have

\[ \alpha^2 \frac{d^2 \phi}{d\xi^2} + i\alpha(2\alpha k - c_g) \frac{d\phi}{d\xi} + (\omega - \alpha k^2) \phi + \beta \phi^3 = 0. \]  

(14)

Setting \( 2\alpha k = c_g \) and \( \omega - \alpha k^2 = -\gamma \), we get

\[ \alpha^2 \frac{d^2 \phi}{d\xi^2} - \gamma \phi + \beta \phi^3 = 0. \]  

(15)

Here we consider the perturbation method [18], letting

\[ \phi = \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots \]  

where \( \epsilon(0 < \epsilon << 1) \) is a small parameter, \( \phi_0 \), \( \phi_1 \), and \( \phi_2 \) represent the zeroth-order, first-order and second-order solutions, respectively.

Substituting (16) into (15), we derive the following systems for the zeroth-order, the first-order and the second-order equations:

\[ \epsilon^0: \quad \alpha^2 \frac{d^2 \phi_0}{d\xi^2} - \gamma \phi_0 + \beta \phi_0^3 = 0, \]  

(17)

\[ \epsilon^1: \quad \alpha^2 \frac{d^2 \phi_1}{d\xi^2} + (3\beta \phi_0^2 - \gamma) \phi_1 = 0, \]  

(18)

and

\[ \epsilon^2: \quad \alpha^2 \frac{d^2 \phi_2}{d\xi^2} + (3\beta \phi_0^2 - \gamma) \phi_2 = -3\beta \phi_0 \phi_1^2. \]  

(19)

The zeroth-order equation (17) can be solved by the Jacobi elliptic sine function expansion method [12,13]; the ansatz solution

\[ u_0 = a_0 + a_1 \text{sn}\xi \]  

(20)
can be assumed.

Substituting (20) into (17), the expansion coefficients \( a_0 \) and \( a_1 \) can be easily determined to be

\[
a_0 = 0, \quad a_1 = \pm \sqrt{\frac{2\gamma}{(1 + m^2)\beta}} m, \quad p^2 = -\frac{\gamma}{(1 + m^2)\alpha},
\]

so the zeroth-order exact solution is

\[
\phi_0 = \pm \sqrt{\frac{2\gamma}{(1 + m^2)\beta}} m \sin \xi.
\]

Substituting the zeroth-order exact solution (22) into the first-order equation (18) yields

\[
\frac{d^2 \phi_1}{d\xi^2} + [(1 + m^2) - 6m^2\sin^2 \xi] \phi_1 = 0.
\]

Obviously this is just a Lame equation like (1) with \( n = 2 \) and \( \lambda = (1 + m^2) \). The Lame equation (1) then reduces to (6), so the solution of (23) is

\[
\phi_1 = AL_2^\beta(\xi) = Acn \xi dn \xi,
\]

where \( A \) is an arbitrary constant, and (24) is the first-order exact perturbed solution of the NLS equation (12).

In order to solve the second-order equation (19), the zeroth-order exact solution (22) and the first-order exact solution (24) have to be substituted into (19). Thus the second-order equation (19) is rewritten as

\[
\frac{d^2 \phi_2}{d\xi^2} + [(1 + m^2) - 6m^2\sin^2 \xi] \phi_2 = \pm 3 \sqrt{-\frac{2\beta A^2}{\alpha}} \frac{m A^2}{p} \sin \xi \cos \xi \sin^2 \xi.
\]

It is obvious that this is an inhomogeneous Lame equation with \( n = 2 \) and \( \lambda = (1 + m^2) \). The solution of its homogeneous equation is just the same as (24); a particular solution for the inhomogeneous terms can be assumed to be of the form

\[
\phi_2 = b_1 \sin \xi + b_3 \sin^3 \xi.
\]

Substituting (26) into (25), we determine the expansion coefficients \( b_1 \) and \( b_3 \) to be

\[
b_1 = \pm \frac{1 + m^2}{4mp} \sqrt{-\frac{2\beta}{\alpha} A^2}, \quad b_3 = \pm \sqrt{-\frac{2\beta}{\alpha} \frac{mA^2}{2p}},
\]

so the second-order exact solution of the NLS equation (12) can be written as

\[
\phi_2 = \pm \frac{1 + m^2}{4mp} \sqrt{-\frac{2\beta}{\alpha} A^2} \sin \xi [1 - \frac{2m^2}{1 + m^2} \sin^2 \xi].
\]
Actually, we know that the zeroth-order equation (17) can be solved by the other Jacobi elliptic function expansion methods [12,13], such as the Jacobi elliptic cosine function expansion method and the Jacobi elliptic function of the third kind expansion method. For the Jacobi elliptic cosine function expansion method, from the zeroth-order equation (17) we can get the zeroth-order exact solution

$$\phi_0 = \pm \sqrt{\frac{2\gamma}{(2m^2 - 1)\beta}} m cn \xi .$$

Substituting the zeroth-order exact solution (29) into the first-order equation (18) yields

$$\frac{d^2 \phi_1}{d \xi^2} + [(1 + 4m^2) - 6m^2 sn^2 \xi] \phi_1 = 0 .$$

Obviously this is just a Lame equation like (1) but with \( n = 2 \) and \( \lambda = (1 + 4m^2) \). The Lame equation (1) then reduces to (8), so the solution of (30) is

$$\phi_1 = AL_2^0(\xi) = A sn \xi dn \xi ,$$

where \( A \) is an arbitrary constant; (31) is another first-order exact perturbed solution of the NLS equation (12).

By substituting the zeroth-order exact solution (29) and the first-order exact solution (31) into (19), the second-order equation (19) can be rewritten as

$$\frac{d^2 \phi_2}{d \xi^2} + [(1 + 4m^2) - 6m^2 sn^2 \xi] \phi_2 = \mp 3 \sqrt{\frac{2\beta m^2 A^2}{\alpha p}} cn \xi sn^2 \xi dn^2 \xi ,$$

from which the second-order exact solution of the NLS equation (12) can be written as

$$\phi_2 = \pm \frac{2m^2 - 1}{4mp} \sqrt{\frac{2\beta}{\alpha}} A^2 cn \xi [1 - \frac{2m^2}{2m^2 - 1} cn^2 \xi] .$$

Similarly, for a Jacobi elliptic function of the third kind expansion method, the zeroth-order exact solution is

$$\phi_0 = \pm \sqrt{\frac{2\gamma}{(2 - m^2)\beta}} dn \xi ,$$

the first-order exact solution is

$$\phi_1 = AL_2^d(\xi) = A sn \xi cn \xi ,$$

and the second-order exact solution of NLS equation (12) can be written as

$$\phi_2 = \pm \frac{2 - m^2}{4m^2 p} \sqrt{\frac{2\beta}{\alpha}} A^2 dn \xi [1 - \frac{2}{2 - m^2} dn^2 \xi] .$$
IV. APPLICATION TO THE COUPLED NLS EQUATIONS

In the above section, we discussed the application of Lame equations, with the condition that \( n = 2 \), to the NLS equation and obtained multi-order exact solutions. Next, we will illustrate the application of the Lame equations under the condition \( n = 2 \) to solve the coupled NLS equations.

The coupled NLS equations read

\[
\begin{align*}
\frac{i}{\partial t} \frac{\partial u}{\partial x} + \alpha \frac{\partial^2 u}{\partial x^2} + \beta |u|^2 u + \delta |v|^2 u &= 0, \\
\frac{i}{\partial t} \frac{\partial v}{\partial x} + \alpha \frac{\partial^2 v}{\partial x^2} + \beta |v|^2 v + \delta |u|^2 v &= 0.
\end{align*}
\]

(37a)  (37b)

We solve them in the following form

\[
\begin{align*}
u &= \phi(x)e^{i(kx-\omega t)}, \\
v &= \psi(x)e^{i(kx-\omega t)}, \\
\xi &= p(x-c_\theta t).
\end{align*}
\]

(38)

Set \( 2\alpha k = c_\theta \) and \( \omega - \alpha k^2 = -\gamma \), then Eq.(37) becomes

\[
\begin{align*}
\alpha p^2 \frac{d^2 \phi}{d\xi^2} - \gamma \phi + \beta \phi^3 + \delta \phi \psi^2 &= 0, \\
\alpha p^2 \frac{d^2 \psi}{d\xi^2} - \gamma \psi + \beta \psi^3 + \delta \phi^2 \psi &= 0.
\end{align*}
\]

(39a)  (39b)

The solutions to (39) can be expanded as a multi-order power series by applying the perturbation method [18], i.e.

\[
\begin{align*}
\phi &= \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots, \\
\psi &= \psi_0 + \epsilon \psi_1 + \epsilon^2 \psi_2 + \cdots,
\end{align*}
\]

(40a)  (40b)

where \( \epsilon(0 < \epsilon << 1) \) is a small parameter. Substituting (39) into (40) leads to the multi-order equations. For example, the first three orders are

\[
\begin{align*}
\alpha p^2 \frac{d^2 \phi_0}{d\xi^2} - \gamma \phi_0 + \beta \phi_0^3 + \delta \phi_0 \psi_0^2 &= 0, \\
\alpha p^2 \frac{d^2 \psi_0}{d\xi^2} - \gamma \psi_0 + \beta \psi_0^3 + \delta \phi_0^2 \psi_0 &= 0, \\
\alpha p^2 \frac{d^2 \phi_1}{d\xi^2} + (3\beta \phi_0^2 - \gamma) \phi_1 + \delta (\psi_0^2 \phi_1 + 2 \phi_0 \psi_0 \psi_1) &= 0,
\end{align*}
\]

(41a)  (41b)  (42a)
\[ \frac{\alpha^2}{d^2} \psi_1 + (3\beta \psi_0^2 - \gamma) \psi_1 + \delta (\phi_0^2 \psi_1 + 2\phi_0 \phi_1 \psi_0) = 0, \quad (42b) \]

and

\[ \frac{\alpha^2}{d^2} \phi_2 + (3\beta \phi_0^2 - \gamma) \phi_2 + \delta (\phi_0^2 \phi_2 + 2\phi_0 \psi_0 \psi_2) = -3\beta \phi_0 \phi_1^2 - 2\delta \phi_1 \psi_0 \psi_1, \quad (43a) \]

\[ \frac{\alpha^2}{d^2} \psi_2 + (3\beta \psi_0^2 - \gamma) \psi_2 + \delta (\phi_0^2 \psi_2 + 2\phi_0 \psi_0 \phi_2) = -3\beta \psi_0 \psi_1^2 - 2\delta \psi_1 \phi_0 \phi_1. \quad (43b) \]

The zeroth-order equation (41) can be solved by the Jacobi elliptic sine function expansion method [12,13] where the ansatz solutions are

\[ \phi_0 = a_0 + a_1 \text{sn} \xi, \quad \psi_0 = b_0 + b_1 \text{sn} \xi. \quad (44) \]

Substituting (44) into (41) yields

\[ \phi_0 = \psi_0 = \pm \sqrt{\frac{2\gamma}{(\beta + \delta)(1 + m^2)}} m \text{sn} \xi. \quad (45) \]

Substitute the zeroth-order solution (45) into the first-order equation (42) to get

\[ \phi_1 = \psi_1 = AL^2_1(\xi) = A \text{cn} \xi \text{dn} \xi. \quad (46) \]

Combining the zeroth-order solution (45) and the first-order exact solution (46) with the second-order equation (43) leads to the second-order exact solution

\[ \phi_2 = \psi_2 = \pm \frac{A^2(1 + m^2)}{4m} \sqrt{\frac{2(1 + m^2)(\beta + \delta)}{\gamma}} \text{sn}^2 \xi \left(1 - \frac{2m^2}{1 + m^2} \text{sn}^2 \xi\right). \quad (47) \]

Just as was done in the above section, we can get two other kinds of multi-order solutions. From the Jacobi elliptic cosine function expansion method the zeroth-order exact solution is

\[ \phi_0 = \psi_0 = \pm \sqrt{\frac{2\gamma}{(2m^2 - 1)(\beta + \delta)}} m \text{cn} \xi, \quad (48) \]

the first-order solution is

\[ \phi_1 = \psi_1 = AL^2_1(\xi) = A \text{sn} \xi \text{dn} \xi, \quad (49) \]

and the second-order exact solution can be written as

\[ \phi_2 = \psi_2 = \pm \frac{2m^2 - 1}{4mp} \sqrt{\frac{2(\beta + \delta)}{\alpha}} A^2 \text{cn} \xi \left[1 - \frac{2m^2}{2m^2 - 1} \text{cn}^2 \xi\right]. \quad (50) \]
For the Jacobi elliptic function of the third kind expansion method, the zeroth-order exact solution is
\[ \phi_0 = \psi_0 = \pm \sqrt{\frac{2\gamma}{(2 - m^2)(\beta + \delta)}} \text{dn} \xi, \] (51)
the first-order exact solution is
\[ \phi_1 = \psi_1 = AL_2^d(\xi) = A\text{sn}\xi\text{cn} \xi, \] (52)
and the second-order exact solution of the NLS equation (12) can be written as
\[ \phi_2 = \psi_2 = \pm \frac{2 - m^2}{4m^4\rho} \sqrt{\frac{2(\beta + \delta)}{\alpha}} A^2\text{dn}[1 - \frac{2}{2 - m^2}\text{dn}^2 \xi]. \] (53)

V. CONCLUSION AND DISCUSSION

In this paper, the Lame equation and the Lame functions are applied to solve the NLS equation and the coupled NLS equations. When the perturbation method and the three kinds of Lame functions \( L_2^a(x), L_2^b(x) \) and \( L_2^c(x) \) are considered, then multi-order solutions are obtained to these nonlinear evolution systems. The results found in this paper are very important for the nonlinear stability analysis of nonlinear waves.

Acknowledgement

This paper is supported by the NSFC (No.40045016 and No.40175016).

References

* Corresponding author. Electronic address: fuzt@pku.edu.cn