On the Equivalence of the Massless DKP Equation and Maxwell Equations in Robertson-Walker Spacetime

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We solve the photon equation and the Maxwell equations separately and show that the Maxwell equations can be obtained as the zero mass limit of the DKP equation in curved spacetimes. Finally, by introducing a curved $6 \times 6$ representation of the matrices in Robertson-Walker spacetime, we show that the Maxwell equations can be written in a Dirac-like equation form.

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I. INTRODUCTION

One of the most important problems in contemporary theoretical physics is the construction of a theory that combines quantum theory and gravity. By using the principle of covariance and the tetrad formalism, with the Tetrode-Weyl-Fock-Ivanenko procedure expanded to include spin transformation quantities, it has been possible to make relativistic equations conform to general relativity. Although general relativistic wave equations may not be important on the atomic scale because of the weakness of gravitational effects, for many astrophysical situations one has to take into account gravitational effects due to their dominant role; for example particle creation by black holes. In addition, it is not meaningful to construct a unified theory of gravitation and quantum theory if the single particle states are not studied carefully. In that context, relativistic particles that satisfy wave equations in cosmological backgrounds are considered, in order to analyze quantum effects in curved space-times. Some restrictions on the exact solutions of the equations are examined to obtain the behavior of the particles.

Generally photons are formulated as the quanta of the Maxwell field in flat and curved space-times. Attempts to write the Maxwell equations as spinor equations, and also to represent photons by a quantum wave equation continue. The properties of the photons states related to the photon wave equation are investigated in these studies. Interactions are introduced as usual with the minimal substitution of the momentum operators. Passage from the classical wave theory of light to quantum mechanics is usually done by writing Maxwell’s equations in Schrödinger form and then replacing the operator $\nabla$ by $\left(\frac{i}{\hbar} \mathbf{p} \right)$, so as to extract the particle aspects of the electromagnetic field. If complex combinations of the electric and magnetic fields are taken as the elements of a three-component spinor, Maxwell’s curl equations can be synthesized into a form similar to that of the Weyl...
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The aim of this paper is to discuss the solutions to this equation in spatially flat Robertson-Walker (RW) spacetime. This metric has a geometrical form which is similar to Minkowski spacetime. Starting from this point, we solve the photon equation and the Maxwell equations separately and show that the Maxwell equations can be obtained as the zero mass limit of the DKP equation in curved spacetimes. Finally, by introducing a curved 6 × 6 representation of the matrices in RW spacetime we show that the Maxwell equations can be written in a Dirac-like equation form.
II. THE MASSLESS DKP EQUATION IN SPATIALLY FLAT ROBERTSON-WALKER SPACE-TIME

In 1997 Ünal, following Barut and Zanghi, proposed a simple classical model of the zitterbewegung of the Dirac electron. By quantizing this simple model he obtained a wave equation for massless spin-1 particles in an external field. He showed that this equation is equivalent to the free space Maxwell equations. For the DKP massive spin-1 equation, the matrices obeying the Kemmer algebra are given as

$$\beta^\mu(x) = \gamma^\mu(x) \otimes I + I \otimes \gamma^\mu(x).$$

(4)

In the technique Ünal employed, he denoted DKP-like matrices as

$$\beta^\mu(x) = \sigma^\mu(x) \otimes I + I \otimes \sigma^\mu(x),$$

(5)

where $\sigma^\mu(x)$ are curved $2 \times 2$ Pauli matrices. This representation of the matrices leads to a spinor which is related to a complex combination of the electric and magnetic fields.

In this paper we study the solutions of the propagation problem in a cosmological model such as

$$ds^2 = C_0^2(t) dt^2 + C_1^2(t) dx^2 + C_2^2(t) dy^2 + C_3^2(t) dz^2.$$ 

(6)

Here for the simplicity we study the solutions of the mDKP equation in spatially flat Robertson-Walker (RW) spacetimes with the line element choice $C_0(t) = 1, C_1(t) = C_2(t) = C_3(t) = a(t).$ This differs from Minkowski spacetime only by an expansion parameter.

The covariant form of the DKP equation is

$$(i\beta^\mu \nabla_\mu - m) \Psi = 0,$$ 

(7)

where the Kemmer matrices in curved space-time are related to those in flat Minkowski space-time by

$$\beta^\mu = \tilde{\epsilon}^\mu_{(i)} \tilde{\beta}^{(i)},$$

(8)

with a tetrad frame that satisfies

$$g_{\mu\nu} = \epsilon_{(i)} \epsilon_{(j)} \eta^{(i)(j)}.$$ 

(9)

The covariant derivative in Eq. (7) is

$$\nabla_\mu = \partial_\mu + \Omega_\mu,$$ 

(10)

with spinorial connections which can be written as

$$\Omega_\mu = \Gamma_\mu \otimes I + I \otimes \Gamma_\mu.$$ 

(11)
where
\[ \Gamma_{\lambda} = -\frac{1}{8}g_{\mu\alpha}\Gamma_{\nu}^{\alpha}\left[\gamma^{\mu}, \gamma^{\nu}\right]. \] (12)

From this point, if the massless case is used, the spin-1 equation reduces to the 4 × 4 massless
DKP equation
\[ \beta^{\mu}\nabla_{\mu}\Psi = 0, \] (13)
where the \( \beta^{\mu} \) are now
\[ \beta^{\mu}(x) = \sigma^{\mu}(x) \otimes I + I \otimes \sigma^{\mu}(x), \] (14)
with \( \sigma^{\mu}(x) = (I, \vec{\sigma}(x)) \) and
\[ \nabla_{\mu} = \partial_{\mu} + \Sigma_{\mu}, \] (15)
where the spinorial connections \( \Sigma_{\mu} \) are given with the limit \( \gamma^{\mu} \rightarrow \sigma^{\mu} \) as
\[ \Sigma_{\mu} = \lim_{\gamma \rightarrow \sigma} (\Gamma_{\mu} \otimes I + I \otimes \Gamma_{\mu}). \] (16)

For the line element given in Eq.(6) the tetrads are
\[ e^{\mu}_{(0)} = \delta^{\mu}_{0}, e^{\mu}_{(i)} = \frac{1}{a(t)}\delta^{\mu}_{i}, i = 1, 2, 3. \] (17)
The curved Dirac matrices are given by \( (\gamma^{\mu}(x) = e^{\mu}_{(\alpha)} \gamma^{(\alpha)}) \)
\[ \gamma^{0} = \tilde{\gamma}^{(0)}, \gamma^{i} = \frac{1}{a(t)}\tilde{\gamma}^{(i)}. \] (18)
The spinorial connections are
\[ \Gamma_{0} = 0, \Gamma_{i} = \frac{\dot{a}}{2}\gamma^{(0)}\gamma^{(i)}. \] (19)

Using a standard representation of the Dirac matrices we obtain the photon equation
\[ \{[2(I \otimes I)\partial_{t} + \frac{1}{a(t)}(\tilde{\sigma}^{(1)} \otimes I + I \otimes \tilde{\sigma}^{(1)})\partial_{x} + \frac{1}{a(t)}(\tilde{\sigma}^{(2)} \otimes I + I \otimes \tilde{\sigma}^{(2)})\partial_{y}
\]
\[ + \frac{1}{a(t)}(\tilde{\sigma}^{(3)} \otimes I + I \otimes \tilde{\sigma}^{(3)})\partial_{z}] + \frac{\dot{a}}{2a(t)}[(\tilde{\sigma}^{(1)} \otimes I + I \otimes \tilde{\sigma}^{(1)})^{2} + \]
\[ + (\tilde{\sigma}^{(2)} \otimes I + I \otimes \tilde{\sigma}^{(2)})^{2} + (\tilde{\sigma}^{(3)} \otimes I + I \otimes \tilde{\sigma}^{(3)})^{2}]\} \Psi = 0, \] (20)
where $\Psi$ is a four-component spinor given by

$$\Psi = \begin{pmatrix} \Psi_A \\ \Psi_B \\ \Psi_C \\ \Psi_D \end{pmatrix}.$$ (21)

After introducing a new time parameter $\eta = \int \frac{1}{a(t)} dt$, Eq. (20) gives four coupled first order differential equations, in terms of the components of the spinor:

$$2 \left( \partial_\eta + \frac{2a(\eta)}{a(\eta)} \right) \Psi_A + \frac{1}{a(t)} \partial_x (\Psi_B + \Psi_C) - \frac{1}{a(t)} i \partial_y (\Psi_B + \Psi_C) + 2 \frac{1}{a(t)} \partial_z \Psi_A = 0,$$ (22)

$$2 \left( \partial_\eta + \frac{2a(\eta)}{a(\eta)} \right) \Psi_B + \frac{1}{a(t)} \partial_x (\Psi_A + \Psi_D) + \frac{1}{a(t)} i \partial_y (\Psi_A - \Psi_D) = 0,$$ (23)

$$2 \left( \partial_\eta + \frac{2a(\eta)}{a(\eta)} \right) \Psi_C + \frac{1}{a(t)} \partial_x (\Psi_A + \Psi_D) + \frac{1}{a(t)} i \partial_y (\Psi_A - \Psi_D) = 0,$$ (24)

$$2 \left( \partial_\eta + \frac{2a(\eta)}{a(\eta)} \right) \Psi_D + \partial_x (\Psi_B + \Psi_C) + i \partial_y (\Psi_B + \Psi_C) - 2 \frac{1}{a(t)} \partial_z \Psi_D = 0.$$ (25)

From Eq. (23) and Eq. (24) it can be seen that $\Psi_B = \Psi_C$. By introducing a function of the form $\Psi = \frac{1}{a(\eta)} \Phi$ we obtain the following equations

$$\partial_\eta \Phi_A + \partial_x \Phi_C - i \partial_y \Phi_C + \partial_z \Phi_A = 0,$$ (26)

$$2 \partial_\eta \Phi_C + \partial_x (\Phi_A + \Phi_D) + i \partial_y (\Phi_A - \Phi_D) = 0,$$ (27)

$$\partial_\eta \Phi_D + \partial_x \Phi_C + i \partial_y \Phi_C - \partial_z \Phi_D = 0.$$ (28)

To separate the variables we write

$$\Phi = \exp [i(-\omega \eta + k_2 y + k_3 z)] f(x),$$ (29)

then we obtain

$$\partial_x + k_2 f_C - i (\omega - k_3) f_A = 0$$ (30)

$$\partial_x - k_2 f_C - i (\omega + k_3) f_D = 0,$$ (31)

$$\partial_x (f_A + f_D) - k_2 (f_A - f_D) - 2i \omega f_C = 0.$$ (32)

Eliminating $f_A$ and $f_D$ from Eqs. (30) and (31), we obtain a second-order differential equation for $f_C$:

$$\partial_x^2 + k_1^2 f_C = 0,$$ (33)
where \( k_1^2 = \omega^2 - k_2^2 - k_3^2 \). This equation has the following solution:

\[
f_C = Ae^{\pm ik_1 x}.
\] (34)

The exact solution of the third (or second) component of the spinor is then

\[
\Psi_C = A \frac{1}{a^2(t)} e^{i \left[ \pm k_1 x + k_2 y + k_3 z - \omega \int \frac{dt}{\sqrt{g}} \right]}. 
\] (35)

III. SOLUTIONS OF THE MAXWELL EQUATIONS IN SPATIALLY FLAT RW SPACETIME

As a particle the photon obeys the massless case of the DKP equation. However, there must be consistency with the corresponding solution of the Maxwell equations. The Maxwell equations in free space are

\[
\frac{1}{\sqrt{-g}} (\sqrt{-g} F_{\mu \nu})_{\nu \rho} = 0
\] (36)

and

\[
F_{\mu \nu, \sigma} + F_{\sigma \mu, \nu} + F_{\nu \sigma, \mu} = 0. 
\] (37)

The contravariant field strengths \( F^{\mu \nu} \) and covariant field strengths \( F_{\mu \nu} \) for the line element given in Eq.(6) are found to be

\[
\begin{align*}
F^{01} &= \frac{1}{a(t)} E^{(1)}, \\
F^{02} &= \frac{1}{a(t)} E^{(2)}, \\
F^{03} &= \frac{1}{a(t)} E^{(3)}, \\
F^{12} &= \frac{1}{a^2(t)} B^{(3)}, \\
F^{13} &= -\frac{1}{a^2(t)} B^{(2)}, \\
F^{23} &= \frac{1}{a^2(t)} B^{(1)},
\end{align*}
\] (38)

\[
\begin{align*}
F_{01} &= -a(t) E^{(1)}, \\
F_{02} &= -a(t) E^{(2)}, \\
F_{03} &= -a(t) E^{(3)}, \\
F_{12} &= a^2(t) B^{(3)}, \\
F_{13} &= -a^2(t) B^{(2)}, \\
F_{23} &= a^2(t) B^{(1)},
\end{align*}
\] (39)

where \( E^i \) and \( B^i \) are the components of the electric and magnetic fields in the local Lorentz frame. In terms of the components, the Maxwell equations can be written as

\[
\partial_x \left( \begin{array}{c} E^{(1)} \\ B^{(1)} \end{array} \right) + \partial_y \left( \begin{array}{c} E^{(2)} \\ B^{(2)} \end{array} \right) + \partial_z \left( \begin{array}{c} E^{(3)} \\ B^{(3)} \end{array} \right) = 0,
\] (44)
If we define a three component complex spinor $G$ as

$$ G = \begin{pmatrix} G^{(1)} \\ G^{(2)} \\ G^{(3)} \end{pmatrix} = \begin{pmatrix} E^{(1)} + iB^{(1)} \\ E^{(2)} + iB^{(2)} \\ E^{(3)} + iB^{(3)} \end{pmatrix}, $$

then the spinor form of the Maxwell equations are found to be

$$ \partial_x G^{(1)} + \partial_y G^{(2)} + \partial_z G^{(3)} = 0, \tag{49} $$

$$ \partial_t G^{(1)} + \frac{1}{a(t)} \partial_y G^{(3)} - \frac{1}{a(t)} i \partial_y G^{(2)} = 0, \tag{50} $$

$$ \partial_t G^{(2)} - \frac{1}{a(t)} i \partial_x G^{(3)} + \frac{1}{a(t)} i \partial_x G^{(1)} = 0, \tag{51} $$

$$ \partial_t G^{(3)} + \frac{1}{a(t)} i \partial_z G^{(2)} - \frac{1}{a(t)} i \partial_z G^{(1)} = 0. \tag{52} $$

By again replacing $\eta = \int \frac{1}{a(t)} dt$ and using $G = \frac{1}{a^2(\eta)} U$ we obtain the following equations

$$ \partial_x U^{(1)} + \partial_y U^{(2)} + \partial_z U^{(3)} = 0, \tag{53} $$

$$ \partial_t U^{(1)} + i \partial_y U^{(3)} - i \partial_x U^{(2)} = 0, \tag{54} $$

$$ \partial_t U^{(2)} - i \partial_x U^{(3)} + i \partial_z U^{(1)} = 0, \tag{55} $$

$$ \partial_t U^{(3)} + i \partial_z U^{(2)} - i \partial_y U^{(1)} = 0. \tag{56} $$

If the following form of the spinor is used to separate the variables

$$ U^{(k)} = \exp \left[ i(-\omega \eta + k_2 y + k_3 z) \right] \chi^{(k)}, \ (k = 1, 2, 3) \tag{57} $$

the components $\chi^1$ and $\chi^2$ can be expressed in terms of $\chi^3$ as

$$ \chi^{(1)} = \frac{i}{-k_3^2 + \omega^2} [k_3 \partial_x + \omega k_2] \chi^{(3)}, \tag{58} $$

$$ \chi^{(2)} = \frac{1}{(k_3^2 - \omega^2)} [\omega \partial_x + k_3 k_2] \chi^{(3)}. \tag{59} $$

From the set of above equations it is found that the second order differential equation for $\chi^{(3)}$ is

$$ \partial_t^2 \chi^{(3)} = -k_1^2 \chi^{(3)}. \tag{60} $$

This is the same as Eq. (33) and the solutions are the same as those for Eqs. (34) and (35).
IV. A DIRAC-TYPE EQUATION FOR THE MAXWELL EQUATIONS

In Section 2 and 3 we solved the photon equation and the Maxwell equations, respectively. We saw that these two different equations give the same second order differential equation for the third component. To relate the other components of these two spinors, we have to use a complex combination of the electric and magnetic fields. This is due to the usage of the 4 \times 4 Kemmer matrices. If we want to avoid this complex combination, it is sufficient to use the 6 \times 6 form of the Kemmer matrices. This can be understood in a different way: The DKP equation describes massive spin-0 and spin-1 particles together for the 16 \times 16 form of the Kemmer matrices that is given as

\[ \beta^\mu = \gamma^\mu \otimes I + I \otimes \gamma^\mu. \]  

(61)

The case in which the spin-0 is eliminated yields a 10 \times 10 representation of the Kemmer matrices, which are used to describe the massive spin-1 particle. When the mass term is dropped, this representation reduces to 6 \times 6 Kemmer matrices. Since the photon is a massless spin-one particle, one would expect that the Maxwell equations can be obtained from this form of the DKP equation. In this form, the photon equation in curved spacetimes can be obtained in a straightforward manner, without invoking the complex combination of the electric and magnetic fields. Now the DKP spinor completely consists of electric and magnetic fields in an ordered form.

If we introduce

\[ \Psi = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}, \]

(62)

then

\[ \beta^0 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \]

\[ \beta^1 = \frac{1}{a(t)} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

(63)

\[ \beta^2 = \frac{1}{a(t)} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ \beta^3 = \frac{1}{a(t)} \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]
then the Maxwell Eqs. (36) and (37) become
\[ \beta^\mu (\partial_\mu + \Sigma_\mu) \Psi = 0, \] (64)
where
\[ \Sigma_0 = \frac{2\dot{a}}{a} \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \]
and \( \overline{\Sigma} = 0. \) (65)

V. CONCLUSION

There have been many attempts in the past to show the symmetries of the Maxwell equations. In this paper we have analyzed the relation between a massless DKP equation and the Maxwell equations in spatially RW spacetimes. We showed that the massless spin-1 particle equation and the free space Maxwell equations have the same solutions.

In sections 2 and 3 we saw that the two different equations give the same second order differential equation for the third component of these spinors. To relate the other components we have to use a complex combination of the electromagnetic fields. This is a consequence of using the \( 4 \times 4 \) DKP-like matrices introduced by Ûnal. If we want to avoid this complex combination it is sufficient to use the \( 6 \times 6 \) matrices we selected in section four. This shows that Maxwell equations can be extracted from a Dirac-like equation for the photon in curved spacetimes. This equation allows us to write the Maxwell equations in matrix form. The matrices introduced here have form similar to the Majarona representation but we do not know which commutation relations they obey. In this way the spinor consists completely of electric and magnetic fields in an ordered form.

References

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