A new approach for studying a phase space description of the photon number distribution of nonclassical states, which is based on wavefunctions defined in the coherent-state quantum phase space representation, is presented. We found that our wavefunctions are similar to Husimi’s function amplitude. We illustrate this approach using the examples of squeezed states and displaced number states, and show that it provides an efficient method for computing the photon number distribution directly in phase space without using the WKB approximation.

PACS numbers: 03.65.-w, 42.50.Dv, 42.50.Ar

I. INTRODUCTION

As is well known, quantum states are described by vectors in a Hilbert space [1]. These vectors can be expanded in the \( x \)-basis or in the \( p \)-basis in order to obtain the usual coordinate or momentum wave functions, respectively. It is also possible to define a phase space wave function, even though this is not unique. The attempts to find a description of the quantum states with a resemblance to the classical picture gave rise to several quasidistribution functions in phase space, namely the Wigner function [2,3], the Husimi-Kano Q-function [4], and the Sudarshan-Glauber P-function [5]. All of them are used to represent quantum states in a phase space-like representation. An advantage of the phase space picture is that one can see the regions, or areas, in phase space that are occupied by the wave function [3], showing that the emergence of diverse quantum phenomena appears as the result of interference of the portions of these functions with itself in some particular areas of the phase space [6, 7, 8].

The concept of interference in phase space rests on the semiclassical interpretation of the quantum mechanical scalar product \( \langle \chi | \psi \rangle \) between two quantum states \( |\chi\rangle \) and \( |\psi\rangle \). There are regions in phase space in which the complex-valued amplitude \( \langle \chi | \psi \rangle \) is non-zero. The picture of the quantum interference of states in phase space was developed as a generalization of the Bohr-Sommerfeld description of quantum states as finite areas [6, 7], with the aid of the phase space distributions [8] or using the WKB approximate wave functions [9]. Indeed, for the harmonic oscillator, the area associated with a number state \( |m\rangle \) is a circular ring centered at the origin of the phase space with interior radius \( r_m = \sqrt{2m} \).
and exterior radius \( r_m^+ = \sqrt{2m + 2} \). The areas associated with coherent states and squeezed states are obtained by displacing, or squeezing and displacing, the one associated with the vacuum state [3]. We can summarize the concept of the interference in phase space by the formula

\[
\langle \chi | \psi \rangle = \sum_i A_{\chi \psi}^i \exp(i \phi_{\chi \psi}^i),
\]

where \( A_{\chi \psi}^i \) and \( \phi_{\chi \psi}^i \) are the \( i \)th component of the overlapping area and their assigned phase, respectively. Using the WKB approximation [9], Dowling et al. obtained an explicit expression for the phases where it is possible to give a geometrical meaning. Some time later, Milburn [10] showed that the interference effect in phase space may also be understood by considering the properties of Husimi’s function,

\[
Q(\alpha) = \frac{1}{\pi} |\langle \alpha | \psi \rangle|^2,
\]

where \( |\alpha \rangle \) is a coherent state. This author notes that the coherent states allow us to rewrite the probability amplitude \( \langle m | \psi \rangle \) as a phase space integral,

\[
\langle m | \psi \rangle = \frac{1}{\pi} \int d^2 \alpha \langle m | \alpha \rangle \langle \alpha | \psi \rangle,
\]

where the functions \( \langle m | \alpha \rangle \) and \( \langle \alpha | \psi \rangle \) could be interpreted as the phase space probability amplitudes for the states \( |m \rangle \) and \( |\psi \rangle \), respectively. Each of these functions is related to the function \( Q(\alpha) \) of the corresponding state,

\[
\langle m | \alpha \rangle = \sqrt{\pi Q_m(\alpha)} \exp(i \phi_m(\alpha)), \\
\langle \alpha | \psi \rangle = \sqrt{\pi Q_\psi(\alpha)} \exp(i \phi_\psi(\alpha)).
\]

Following this line of thought, in this paper we calculate the photon number distributions for squeezed and displaced number states in the phase space representation introduced by Torres-Vega and Frederick [11–15]. Husimi’s densities can be obtained as the squared modulus of these functions. This approach allows us to calculate the probability amplitude integral (2) without resorting to the WKB approximation and to find the phases \( \phi_m(\alpha) \) and \( \phi_\psi(\alpha) \) that define Equation (3). This result is interesting in particular for the squeezed states which have, for high squeezing, probabilities of even and odd photon numbers oscillating independently. These oscillations are similar to the ones that appear for squeezed states, which have been discussed thoroughly using the phase space description [7].

II. COHERENT STATE REPRESENTATION

The Wigner function is always well defined for any quantum state and widely used for the calculation of densities and expectation values in phase space [2, 3]. Its disadvantage
FIG. 1: Phase space distributions for the DNS corresponding to Equation (6), centered at \((q_o, p_o) = (3, 0)\) with \(n = 5\) and \(\alpha = 0.0\).

FIG. 2: Phase space distributions for the SDS corresponding to Equation (9), centered at \((q_o, p_o) = (0.3590886913, 0)\) with \(\alpha = -0.5\tanh(3.0)\).

is the difficulty of solving realistic problems in terms of the Wigner function, due to the complexity of the corresponding evolution equation. In order to overcome this difficulty, we use the coherent-state representation (CSR) [11–14]. It is called this because it coincides with the totality of the coherent state representations for the Heisenberg-Weyl group [14]. We find this representation particularly useful for our purpose, because it allows for the analysis of quantum dynamics in a phase space in terms of wave functions. It provides us with a way for calculating expectation values as is done in the Schrödinger representation, and the calculations are easier to do as compared with other phase space functions [3]. On the other hand the squared magnitude of the phase space wave function is analogous to the Husimi density [4, 14], which is a tool used to compare classical and quantum dynamics. In the CSR there is a two-variable function \(\psi(q, p) = \langle q, p | \psi \rangle\) defined on the real “phase space” \(\Gamma = (q, p)\). This function is a solution of a Schrödinger like equation and the quantity \(|\psi(\Gamma)|^2 \equiv \psi^*(\Gamma)\psi(\Gamma)\) can be used as a probability density with \(\psi^*(\Gamma) = \langle \psi | \Gamma \rangle = \langle \Gamma | \psi \rangle^*\).
FIG. 3: Photon number distribution for a DNS corresponding to the Equation (6), with \( q_0 = 10.1\sqrt{2} \), \( p_0 = 0.0 \), \( \alpha = 0.0 \), and \( n = 3 \), as was pointed out in Reference [26]. The symbols “+” and “×” represent the theoretical and numerical computation, respectively.

This definition ensures that the quantum density \(|\psi(\Gamma)|^2\) is a nonnegative quantity in phase space. The calculation of the expectation value of the operator \( \hat{A} \) is carried out as usual, i.e.,

\[
\langle \hat{A} \rangle = \int d\Gamma \psi^* (\Gamma) \hat{A} \psi (\Gamma),
\]

where the integration is performed over the whole phase space with \( d\Gamma = dqdp \). The real and imaginary parts of the \( \alpha \) of previous sections now play the role of \( q \) and \( p \). The operators associated to the momentum \( \hat{P} \) and coordinate \( \hat{Q} \) are given by \( \hat{P} = (p/2 - i\hbar \partial/\partial q) \) and \( \hat{Q} = (q/2 + i\hbar \partial/\partial p) \). These operators obey the relation \( [\hat{Q}, \hat{P}] = i\hbar \). The annihilation and creation operators in the CSR can be written in terms of the operators \( \hat{Q} \) and \( \hat{P} \), previously defined. The effects of the operators \( \exp(i\eta \hat{Q}/\hbar) \) and \( \exp(i\eta \hat{P}/\hbar) \) when they are applied on the basis vector \( |\Gamma\rangle \) are given by

\[
\exp(i\eta \hat{Q}/\hbar)|p, q\rangle = \exp(i\eta q/2\hbar)|p - \eta, q\rangle,
\]
\[
\exp(i\eta \hat{P}/\hbar)|p, q\rangle = \exp(i\eta p/2\hbar)|p, q + \eta\rangle.
\]

For simplicity in the following, we will consider \( \hbar = m = \omega = 1 \) and the harmonic oscillator system. The number states (NS) in the CSR are described [13] by the following equation:

\[
\langle \Gamma|n \rangle = N H_n(\Gamma; \alpha) \exp \left( -\frac{1}{2} \gamma q^2 - \frac{1}{2} \phi p^2 - i\alpha pq \right),
\]

where \( N \) is a normalization constant, \( \gamma = \frac{1}{2} + \alpha \), \( \phi = \frac{1}{2} - \alpha \), and \( \alpha \) is a complex-valued parameter with \(|\alpha| < \frac{1}{2}\). As it happens with the coherent state representation, these
are interpolation functions between the coordinate \( \alpha = \frac{1}{2} \) and momentum \( \alpha = -\frac{1}{2} \) representation, and \( \alpha = 0 \) gives equal weight to both \( q \) and \( p \). \( H_n(\Gamma; \alpha) \) are a set of orthogonal polynomials defined in phase space, which satisfy the recursion relation

\[
H_{n+1}(\Gamma; \alpha) = 2u(\Gamma; \alpha)H_n(\Gamma; \alpha) - 4n\alpha H_{n-1}(\Gamma; \alpha),
\]

where \( u(\Gamma; \alpha) = \gamma q - i\phi p \). Some of these polynomials are

\[
\begin{align*}
H_0(\Gamma; \alpha) &= 1, \\
H_1(\Gamma; \alpha) &= 2u(\Gamma; \alpha), \\
H_2(\Gamma; \alpha) &= 4u^2(\Gamma; \alpha) - 4\alpha, \\
H_3(\Gamma; \alpha) &= 8u^3(\Gamma; \alpha) - 24u(\Gamma; \alpha)\alpha.
\end{align*}
\]

Note the close resemblance to the usual Hermite polynomials; actually they become them when \( \alpha = 1/2 \). In accordance with Nieto [17], the displaced number states DNS are found by applying the displacement operator \( \hat{D}(\beta) \) [16] to the NS, i.e., \([n, \beta] = \hat{D}(\beta)[n] \), where

\[
\hat{D}(\beta) = \exp(\beta \hat{a}^\dagger - \beta^* \hat{a}) = \exp\left(-i\beta_0 \hat{P} + i\beta_0 \hat{Q}\right).
\]

Here \( \beta \) is the coherence parameter \( \beta = (q_0 + i\phi_0)/\sqrt{2} \). Using the above definition as well as the Baker-Campbell-Hausdorff relation (BCH) [18] and Eqs. (5), we find that the DNS in quantum phase space are [19]

\[
\langle \Gamma, q_0, p_0 | n \rangle = \mathcal{N} H_n(\Gamma, q_0, p_0; \alpha) \exp\left[-\frac{1}{2}\gamma(q - q_0)^2 - \frac{1}{2}\phi(p - p_0)^2 + i(\gamma p_0 q - \phi p_{0,0} - \alpha p q + \phi p_{0,0})\right].
\]
We again have that $H_n(\Gamma, q_o, p_o; \alpha)$ are a set of orthogonal polynomials, but they satisfy the modified recursion relation

$$H_{n+1}(\Gamma, q_o, p_o; \alpha) = 2u(\Gamma, q_o, p_o; \alpha)H_n(\Gamma, q_o, p_o; \alpha) - 4n\alpha H_{n-1}(\Gamma, q_o, p_o; \alpha),$$

where $u(\Gamma, q_o, p_o; \alpha) = \gamma(q - q_o) - i\phi(p - p_o)$. Some of these polynomials are

$$H_0(\Gamma, q_o, p_o; \alpha) = 1,$$

$$H_1(\Gamma, q_o, p_o; \alpha) = 2u(\Gamma, q_o, p_o; \alpha),$$

$$H_2(\Gamma, q_o, p_o; \alpha) = 4u^2(\Gamma, q_o, p_o; \alpha) - 4\alpha,$$

$$H_3(\Gamma, q_o, p_o; \alpha) = 8u^3(\Gamma, q_o, p_o; \alpha) - 24u(\Gamma, q_o, p_o; \alpha)\alpha.$$

The case $\alpha = 0$ is particularly interesting, since it allows us to make contact with classical mechanics. For this value of $\alpha$ the quantum probability density, as given by the squared magnitude of Equation (6), is

$$|\psi_n(\Gamma; \alpha = 0)|^2 = \frac{1}{2\pi n!} \left[ \frac{1}{2}(q - q_o)^2 + \frac{1}{2}(p - p_o)^2 \right]^n \exp \left[ -\frac{1}{2}(q - q_o)^2 - \frac{1}{2}(p - p_o)^2 \right].$$

(7)

This is a function of the classical Hamiltonian for the displaced harmonic oscillator, $H(\Gamma) = (p - p_o)^2/2 + (q - q_o)^2/2$, and hence is also a stationary solution to the classical Liouville equation [13]. The Q-function of a number state [20] is

$$Q(\delta, \beta) = \frac{1}{\pi} \frac{|\delta - \beta|^{2n}}{n!} \exp(-|\delta - \beta|^2),$$

(8)

where $\delta = (q + ip)/\sqrt{2}$ and $\beta = (q_o + ip_o)/\sqrt{2}$. Note that (7) is a half of the number state Husimi’s function (8). The squeezed-state with complex squeezing parameter $\xi = \eta \exp(i\theta)$ can be obtained from the vacuum state (NS with $n = 0$) by replacing the coordinate $q$ by $qS$ and the momentum $p$ by $p/S$ [21] with $S^2 = (1/2 - \alpha)/(1/2 + \alpha)$ and $\alpha = -1/2 \tanh(2\eta) \exp(-i\theta)$, where $|\alpha| < \frac{1}{2}$. By applying the above canonical transformation onto the vacuum state followed by the displacement operator, we obtain the squeezed displaced states (SDS)

$$|\langle \Gamma, q_o, p_o, \xi | 0 \rangle| = \left( \frac{\sqrt{1 - |\alpha|^2}}{\pi} \right)^{1/2} \exp \left[ -\frac{1}{2}\phi(q - q_o)^2 - \frac{1}{2}(p - p_o)^2 + i(\gamma p_o q - \phi p o - \alpha p q + \phi p o q) \right],$$

(9)

and its quantum probability density

$$|\langle \Gamma, q_o, p_o, \xi | 0 \rangle|^2 = \sqrt{\frac{1}{4} - |\alpha|^2} \frac{S}{\pi S^2 + 1} \exp \left[ -\phi(q - q_o)^2 - \gamma(p - p_o)^2 \right].$$

(10)

The Q-function of a SDS [20] reads

$$Q(\delta, \beta) = \frac{2}{\pi} \frac{S}{S^2 + 1} \exp \left[ -\frac{2S^2}{S^2 + 1}(\delta_r - \beta_r)^2 - \frac{2}{S^2 + 1}(\delta_i - \beta_i)^2 \right],$$

(11)
where \( \delta_r = q/\sqrt{2} \), \( \beta_r = q_o/\sqrt{2} \), \( \delta_i = p/\sqrt{2} \), and \( \beta_i = p_o/\sqrt{2} \). Then we can make the identification
\[
\frac{S^2}{S^2 + 1} = \phi, \quad \frac{1}{S^2 + 1} = \gamma, \quad \frac{S}{S^2 + 1} = \sqrt{\frac{1}{4} - |\alpha|^2}.
\]
Note that the density (11) is twice the density of Equation (10). We have obtained that both Equations (6) and (9) can be used to fulfill Milburn’s hypothesis proposed in Equation (3), because the square of their wave amplitudes coincide with the Husimi densities. Clearly, the phases \( \phi_m(\alpha) \) and \( \phi_{\psi}(\alpha) \) can be explicitly obtained. Having this in mind, below we discuss why the wave functions (6) and (9) can also be addressed as Husimi’s wave functions. As was mentioned earlier, the CSR coincides with the totality of coherent-state representations for the Heisenberg-Weyl group [14], and this fact leads to ambiguities when we want to solve the stationary Schrödinger equation in phase space. In certain cases these ambiguities were removed by supplying the Schrödinger equation with an additional differential equation [14] for a minimum uncertainty state fiducial vector, which plays a prominent and crucial role in the CSRs. This vector must be present when images of CSR densities are interpreted as Husimi or Husimi-like functions. In Refs. [13] and [14] it was found that, for the harmonic oscillator in the CSR, the fiducial vector is equal to the ground state. This is the case for any phase space representation eigenstate of the harmonic oscillator. The Husimi function may be obtained from a given wave function in two ways. It is a Gaussian smoothing of the Wigner function [22], and it also is the expected value of the density operator in the basis of coherent states [23]. We will use the later. The Husimi density in phase space \((q,p)\) is defined as
\[
\rho(q,p) = \left| \int dq' \Phi^{(s)*}_{p,q}(q') \psi(q') \right|^2,
\]
which is given in terms of the projection of the wavefunction \( \psi(q) \) onto the coherent states [23],
\[
\Phi^{(s)}_{p,q}(q') = \langle q' | p, q; (s) \rangle = \left( \frac{s}{\pi \hbar} \right)^{1/4} \exp \left[ -\frac{s}{2\hbar} (q - q')^2 + i \frac{p}{\hbar} \frac{q'}{2} \right],
\]
where \( s \) is a squeezing-parameter. Observe that
\[
i\hbar \frac{\partial}{\partial q} \Phi^{(s)*}_{p,q}(q') = \left[ \frac{p}{2} - i\hbar \frac{\partial}{\partial q} \right] \Phi^{(s)*}_{p,q}(q'), \quad q' \Phi^{(s)*}_{p,q}(q') = \left[ \frac{q}{2} + i\hbar \frac{\partial}{\partial p} \right] \Phi^{(s)*}_{p,q}(q').
\]
If the potential \( V \) can be expanded in a power series, the Schrödinger equation in position-space may be written as
\[
\frac{1}{2m} (-i\hbar)^2 \frac{\partial^2}{\partial q^2} \psi(q') + \sum_{n=0}^{\infty} V_n q^n \psi(q') = i\hbar \frac{\partial}{\partial t} \psi(q').
\]
The Schrödinger equation in phase space is obtained by taking the inner product of the above equation with the coherent state:

\[
\frac{1}{2m} \left(-i\hbar\right)^2 \int dq' \Phi_P(q') \frac{\partial^2}{\partial q'^2} \psi(q') + \sum_{n=0}^{\infty} V_n \int dq' \Phi_P(q') q^n \psi(q') = \int dq' \Phi_P(q') \psi(q').
\]

Then, by using the equalities (13), we find that the phase space Schrödinger equation is

\[
\frac{1}{2m} \left(\frac{p}{2} - i\hbar \frac{\partial}{\partial q}\right)^2 \psi(q,p) + V \left(\frac{q}{2} + i\hbar \frac{\partial}{\partial p}\right) \psi(q,p) = i\hbar \frac{\partial}{\partial t} \psi(q,p), \tag{14}
\]

as was pointed out in Reference 14, where \(\psi(q,p)\) is the phase space wave function defined by

\[
\psi(q,p) = \int dq' \Phi_P(q') \psi(q').
\]

Now we turn to the computation of the photon distribution of the squeezed states within the CSR. It is given by

\[
p(n) = |\langle n|\psi\rangle|^2. \tag{15}
\]

Here, the phase space representation of the ket \(|n\rangle\) will be that of Equation (6), with \(\alpha = 0\) and \(q_o = p_o = 0\). The phase space representation of the ket \(|\psi\rangle\) corresponds to the squeezed displaced state (9) with real \(\alpha \neq 0\), \(q_o \neq 0\), and \(p_o = 0\). Then the probability amplitude, expressed in terms of the phase space wave functions, takes the form

\[
\langle n|\psi\rangle = \frac{1}{\sqrt{2}^{n+1} n!} \left(\frac{1 - \alpha^2}{\pi}\right)^{1/2} \exp \left[-\frac{1}{2} \left(\frac{1 - \alpha^2}{\pi}\right) q_o^2\right] \int_{-\infty}^{\infty} dq \exp \left[-\frac{1}{2} (1 - \alpha) q^2\right] n! \int_{-\infty}^{\infty} dp \exp \left\{ -\frac{1}{2} (1 + \alpha) p^2 + i \left(\frac{1}{2} - \alpha\right) q_o p + \alpha q p\right\}.
\]

In this expression we can use the integral representation of the Hermite polynomial \(H_n(x)\)

\[
H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + it)^n e^{-t^2} dt,
\]

and the relation [25]

\[
\int_{-\infty}^{\infty} \exp \left[-\frac{(x - y)^2}{2u}\right] H_n(x) dx = (2\pi u)^{1/4} (1 - 2u)^{n/2} H_n \left[ y(1 - 2u)^{-1/2} \right],
\]
and write an explicit formula for Equation (15) as

\[
p(n) = \frac{1}{2^{n+1} \pi n!} \frac{\sqrt{\frac{4}{\pi} - \alpha^2}}{n} \left( \frac{2}{1 + \alpha} \right)^{n+1} \frac{\pi}{2^{2n}} \frac{2(1 + \alpha)}{(1 + 2\alpha)^2} \exp \left[ -\left( \frac{1}{2} - \alpha \right) q_o^2 \right] \times \left| H_n \left( \frac{1}{\sqrt{2}} \alpha q_o \sqrt{-2\alpha} \right) \right|^2.
\]

By using \( \alpha = -\frac{1}{2} \tanh 2\eta \) in the above relation we obtain

\[
p(n) = \frac{\tanh^n 2\eta}{n!2^n \cosh 2\eta} \exp \left[ -\frac{q_o^2}{2} (1 + \tanh 2\eta) \right] \left| H_n \left[ \frac{q_o}{\sqrt{2}} (\cosh 2\eta + \sinh 2\eta) \right] \right|^2,
\]

and by noting that \((\cosh 2\eta + \sinh 2\eta)^2(1 - \tanh 2\eta) = 1 + \tanh 2\eta\), we obtain

\[
p(n) = \frac{\tanh^n 2\eta}{n!2^n \cosh 2\eta} \exp \left[ -\frac{q_o^2}{2} (1 + \tanh 2\eta) \right] \left| H_n \left[ \frac{q_o}{\sqrt{2}} (\cosh 2\eta + \sinh 2\eta) \right] \right|^2. \tag{16}
\]

One way of obtaining a squeezed state is by applying the displacement operator and then the squeeze operator, \( |\xi, \beta\rangle = \hat{S}(\xi) \hat{D}(\beta) |0\rangle \), as Mundarain and Stephany did in Ref. [26]. Another way is to apply the squeeze operator first and then to apply the displacement operator second, as we have just done. In general \( \hat{S}(\xi) \hat{D}(q_o, p_o) = \hat{D}(q_o', p_o') \hat{S}(\xi) \) when

\[
q_o' = q_o (\cosh 2\eta + \cos \theta \sinh 2\eta) + p_o \sin \theta \sinh 2\eta,
\]

\[
p_o' = q_o \sin \theta \sinh 2\eta + p_o (\cosh 2\eta - \cos \theta \sinh 2\eta).
\]

For one case \( \theta = 0 \) and \( p_o = 0 \), and using the above relation one finds that Eq. (16) becomes

\[
p(n) = \frac{\tanh^n 2\eta}{n!2^n \cosh 2\eta} \exp \left[ -\frac{q_o^2}{2} (\tanh 2\eta - 1) \right] \left| H_n \left[ \frac{q_o}{\sqrt{2}} \frac{1}{\sqrt{2} \sinh 2\eta \cosh 2\eta} \right] \right|^2,
\]

Now, since \( \beta = q_o^{\prime}/\sqrt{2} \), the probability amplitude may be written as

\[
p(n) = \frac{\tanh^n 2\eta}{n!2^n \cosh 2\eta} \exp \left[ \beta^2 (\tanh 2\eta - 1) \right] \left| H_n \left[ \frac{\beta}{\sqrt{2} \sinh 2\eta \cosh 2\eta} \right] \right|^2, \tag{17}
\]

which is an identical relation to that used by Mundarain and Stephany [26] and a particular case obtained by Yuen [27]. Similarly, the photon distribution of the DNS \( \langle \Gamma, q_o, p_o | n \rangle \) within the CSR is given by

\[
p(n, m) = \frac{n!}{m!} \exp \left[ -(q_o^2 + p_o^2)/2 \right] \left[ \frac{1}{2} (q_o^2 + p_o^2) \right]^{m-n} \left[ L_n^{m-n}(q_o^2 + p_o^2)/2 \right]^2, \tag{18}
\]
when $m \geq n$, and

$$p(n, m) = \frac{m!}{n!} \exp \left[ -\frac{1}{2} (q_o^2 + p_o^2) \right] \left[ \frac{1}{2} (q_o^2 + p_o^2) \right]^{n-m} \left[ L_m^{n-m}((q_o^2 + p_o^2)/2) \right]^{2}.$$  \hspace{1cm} (19)

when $n \geq m$, where $L_m^{n-m}(x)$ is the associated Laguerre polynomial.

### III. DISCUSSION

The expression of $\hat{Q}$ and $\hat{P}$ in the CSR offers us a direct way to generate information about quantum mechanics that can easily be compared with classical mechanics. As long as the potential function is expressible as a power series in $q$, we are able to write down an effective Schrödinger equation for the system, which is applicable in phase space and which maintains the canonical commutation relation $[\hat{Q}, \hat{P}] = i \hat{I}$. In this representation of the Schrödinger equation, one can determine the exact time evolution of a nonstationary state completely in phase space [12]. However, one does not necessarily have to solve a partial differential equation in phase space in order to find the evolution of a phase space wave function. One alternative is to generate the desired state in position space and then convert it to the phase space representation by means of the Husimi transform (12). Note that there is a dependency of the Husimi function on the value of $\alpha$, and the squeezing-parameter $s$ is also a free parameter, both of which affect the shape of the phase space quantum density.

The behavior of photon statistics and interference in phase space has been extensively studied previously [7–9, 20, 28], but it would also be interesting to study the squeezed number states in phase space [29]. In order to clarify fundamental concepts applied to this interesting subject, we have used the algorithm described in Reference [12]. It is also suitable for the calculation of expectation values; one can numerically calculate the mean photon number, $\langle\hat{N}\rangle$, for displaced number states (Eq. (6)) in order to adjust the size of the rectangular grid, that is necessary for calculating the correct value of integral (4). We have defined $\hat{N} = \frac{1}{2}(\hat{P}^2 + \hat{Q}^2 - 1)$. The squared modulus of the phase space probability amplitude for the displaced number state (Eq. (7)) $|\langle \Gamma, q_o, p_o|n \rangle|^2$ is shown in Figure 1, with $q_o = 3.0$, $p_o = 0.0$, $\alpha = 0.0$, and $n = 5$. This function allows us to associate the corresponding $n$th number state with the Poisson Band, i.e, a circular ring centered at the origin and located between the radii $\sqrt{2n-1}$ and $\sqrt{2n+1}$. In Figure 2 we show the phase space probability density for the displaced squeezed state $|\langle \Gamma, q_o, p_o, \xi |0 \rangle|^2$ (Equation (10)), with $q_o = 0.3590886913$, $p_o = 0.0$, $\alpha = 0.0$, and $n = 3$, as was used by Mundarain and Stephany [26]. The symbols “+” and “×” represent the theoretical (Equations (18) and (19)) and the numerical computations, respectively. In order to show our symbols we have left out a few. The numerical computation shows a very good qualitative and quantitative behavior. In Figure 4 we show the photon number distribution of SDS corresponding to Equation (9), with $q_o = 0.3590886913$, $p_o = 0.0$, and $\alpha = -0.5 \tanh(3.0)$, as in Reference [26]. The symbols “+” and “×” represent the theoretical (Equation (17)) and numerical computation, respectively.
Again, the numerical computation shows a very good qualitative and quantitative behavior. This result is interesting because of the high squeezing. We have the probabilities of even and odd photon numbers oscillating independently.

IV. CONCLUSIONS

In this work we have shown that the corresponding amplitude modulus of the \( n \)th number state and the displaced squeezed state wave functions defined in the CSR are related to the Q-function phase space. So, our coherent state representation provides us with another way to calculate the photon statistics of a displaced squeezed state. There are three different kinds of the phase space: the Planck-Bohr-Sommerfeld, the Wigner, and the Q-function phase spaces; thus three different approaches for calculating quantities like the photon statistics of a squeezed state. The three approaches yield identical results. As was mentioned above, one could have generated the phase space functions from a wave function in position space by means of the Husimi transform, as was done in reference [28] by Vogel and Schleich, in fact they have shown that the Q-function phase space approach rests on the representation of the \( n \)th number state as a continuous superposition of coherent states along a circle in phase space. On the other hand, Smith [15] has shown how the quantum structure envisioned by Torres-Vega and Frederick is related with the usual formalism of coherent states. The main goal of this paper was to show that the analysis of quantum dynamics in a phase space is possible in terms of wave functions, as well as providing an alternative way of calculating expectation values in a similar way as is done in the Schrödinger representation. This representation allows for the analysis of classical and quantum dynamics in the same phase space. Finally, we would like to mention that a similar approach can be also applied to compute the photon number distribution for squeezed number states [19, 29].

Acknowledgments

We thank the referee for important observations on an earlier version of this paper. The authors would like to thank Dr. Go. Torres-Vega for careful reading of the manuscript and enlightening comments. A. Zúñiga S. thanks the Departamento de Estado Sólido, Instituto de Física, Universidad Nacional Autónoma de México for the hospitality extended to him.

References


