A Maximum Beam Intersection Method for the Focusing Property Analysis of Flat Lenses

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This study presents a statistical method for the focusing point analysis of flat lenses. The proposed method, referred to as the Maximum Beam Intersection method (MBI), analyzes the beam intersection density distribution on the focal axis of a flat lens by means of ray trace analysis. The MBI method not only provides an estimation of the focusing point location but also allows an assessment of the quality of the focusing points. The analysis demonstrates that the ideal imaging effect, obtained in the case of $n = -1$, has the highest beam intersection density. Results of the MBI method are in agreement with the results of the conventional analytical calculation and the flat lens simulation results.

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I. INTRODUCTION

Artificial crystals are made of periodic scatterer arrays embedded in a host media. Recent studies have shown the wave propagation in crystal structures to have several useful assets, such as band gaps, negative refraction, highly efficient wave guiding, and sub-wavelength focusing [1–20]. An essential feature of crystal structures is the scalability of the frequency characteristics in a wide frequency range, from low-audible frequencies to the terahertz range [6, 12]. This scalability enables a straightforward implementation of new wave-control devices by scaling existing crystal configurations according to the working frequencies. Due to the technological potential, the amazing properties of crystal structures have been of interest to researchers over the last two decades, and the negative refractive crystal structures hold an important place in metamaterial science.

The negative refraction effect observed in crystal structures enables the implementation of flat lenses, which contributes to the development of superior imaging and wave forming applications [1–4, 6, 7, 9, 16–19]. Analyses of the focusing properties, therefore, have become more substantial in both theoretical and practical works. Several methods were used for the analysis of the focusing properties in the flat lens: (i) analysis based on band structure and equifrequency counter (EFC) characteristics, generally obtained by the plane wave expansion (PWE) method [10–12, 16, 19], (ii) analysis based on numerical simulations, such as finite difference time domain (FDTD) methods [12–15, 19] and multiple scattering theory (MST) methods [16, 17], and (iii) analytical methods based on ray trace

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In RT analysis, when the crystal slab exhibits a negative effective refractive index (ERI) \((n < 0)\), a beam intersection on the focal axis is assumed to be a wave focalization. In this sense, Qiu et al. developed a geometrical analysis, applying a near-axis approximation to the RT analysis \([18]\). This geometrical analysis was used successfully to estimate the focal length of the negative refractive sonic flat lenses \([18, 19]\). However, it does not consider the beam intersection point distribution on the focal axis.

This paper essentially presents a statistical method, based on histogram analysis of beam intersection points on the focal axis, as an extension of RT analysis. The proposed method calculates the distribution of the beam intersection points on the focal axis of flat lenses. Thus, the MBI method not only provides an estimation of the focusing point location but also allows an assessment of the intensity of the focusing points.

In the paper, a triangular lattice sonic crystal slab with a lattice constant of 15 mm and a diameter of 5 mm is used as an acoustic flat lens and inspected for the wave focusing effect. This sonic crystal slab is composed of aluminum circular bar arrays in air. Figures 1 (a) and (b) show the band structure and EFS characteristics of the second band of the sonic crystal. The band structure and equifrequency surface (EFS) characteristics of the crystals have a convex form around the \(\Gamma\) high symmetry point. This implies that the wave vectors \((\vec{k})\) and the group velocity \((\vec{v}_g = \nabla_{\vec{k}} w(\vec{k}))\) are opposite \((\vec{k} \cdot \vec{v}_g < 0)\), and that the crystal slab exhibits negative ERI and behaves as an acoustic metamaterial in the second band.

The negative ERI estimation of the analyzed triangular sonic crystal in the second band was calculated by using \(n = -w/(c. |\vec{k}|)\) and is plotted in Figure 1(c). Figure 1 reveals that this sonic crystal slab exhibits enough large negative ERI in the frequency range of 14–18 kHz and works as an acoustic flat lens.

In the following sections, the proposed maximum beam intersection (MBI) method is explained and its results obtained for the analyzed sonic flat lens are compared with results of the analytical method proposed by Qiu et al. \([18]\). In advance, a brief discussion is given for the ideal imaging effect \((n = -1)\).

**II. MAXIMUM BEAM INTERSECTION METHOD**

It would be helpful to see the RT analysis of a negative refractive flat lens in Figure 2(a), and the FDTD simulation results demonstrating the acoustic wave focusing effect of the analyzed sonic crystal slab in Figure 2(b) and (c). Beams are emitted from a point source with a \(D_1\) distance from the flat lens. After negative refractions of the beams at the interfaces of the crystal slab, they intersect at the \(P_2\) point inside the crystal and the \(P_3\) point outside the crystal. The distance of the \(P_3\) intersection point from the flat lens is represented by \(f\). By doing a geometrical investigation, the \(f\) distance with respect to negative ERI can be expressed as

\[
f = \frac{\tan \alpha_c}{\tan \alpha_h} d - D_1,
\]  

\((1)\)
FIG. 1: (a) The band structure characteristic of a sonic crystal by means of the PWE method, (b) EFS of the second band, and (c) ERI estimation of the sonic crystal.

\[ n = \frac{\sin \alpha_h}{\sin \alpha_c}. \]  

(2)

Here, \( \alpha_h \) and \( \alpha_c \) are the angles of wave beams with respect to the interface normal of the flat lens. The parameter \( n \) is the ERI of the flat lens, defined by Equation (2). (A derivation of Equation (1) is given in the Appendix.)

The focusing point of a lens is the point on the focal axis where the intensity of the waves reaches a maximum value. Hence, the intensification of beam intersections at a point of the focal axis is an indication of the focusing point in the RT analysis. Figure 3 presents beam intersection points showing the focusing effect, depending on the value of \( n \). In the case of \( n = -1 \), all beams intersect at a single point on the focal axis, which is called “ideal imaging” [18]; it provides the best quality and the highest intensity in wave focalization.
In the case of $n \neq -1$, several intersection points spread over the focal axis, and therefore, this spread reduces the quality of wave focalization in terms of intensification. In all cases, the focusing point will be the region where the beam intersections are mostly intensified. For this reason, this approach to detect the focusing points is referred to as the Maximum Beam Intersection method.

FIG. 2: (a) Simple RT analysis of a flat lens [18]. (b) Pressure field image of the wave focusing effect, and (c) intensity distribution of the focal point from the FDTD simulation of an acoustic flat lens at the frequency of 14650 Hz ($n \cong -1$).
In the MBI method, a histogram analysis of the beam intersection points residing on the focal axis of the sonic crystal slab is conducted for a given range of negative ERIs. According to the proposed method, for each value of $n$, the $f$ distances for each beam intersection point are computed using Equation (1) for all $\alpha_h$ angles in the range of $[0, \alpha_{\text{max}}]$. $\alpha_{\text{max}}$ is the maximum angle of the refracted beam, and it can be found according to the critical angle of Snell’s Law by using $\alpha_{\text{crit}} = \sin^{-1}(n)$. In these calculations, the $\alpha_c$ parameter is calculated by using $\alpha_c = \sin^{-1}(\sin\alpha_h/n)$, which is derived from Equation (2). After calculating the histogram of $f$ distances for each value of $n$, as demonstrated in Figure 4, the $f$ distance having the maximum value in the histogram is selected as the focal length $f_o$. The histogram value at the selected $f_o$, denoted by $N_o$, gives the number of beam intersections supporting the focal length $f_o$. After obtaining $f_o$ and $N_o$ for all negative $n$ values in a predefined range, the $f_o$ and $N_o$ characteristics are plotted versus $n$, as depicted in Figure 5. (A programming algorithm for the computation of the MBI method is presented in the Appendix).

Basic calculation steps for the MBI method can be summarized as follows:

**Step 1:** Calculate $f(n, \alpha_h)$ by using Equation (1) for a predetermined range of $n$ and $\alpha_h \in (0^\circ, \alpha_{\text{max}})$ in appropriate steps. In these calculations, use $\alpha_c = \sin^{-1}(\sin\alpha_h/n)$.

**Step 2:** For each $n$, form the $f_n$ set, defined as $f_n = \{f(i, \alpha_h), i = n \lor \alpha_h \in (0, \alpha_{\text{max}})\}$

**Step 3:** Calculate the histogram in each $f_n$ and assign the maximum value of the
FIG. 4: Beam intersection histograms for various values of $n$. (a) $f_{-0.7}$, (b) $f_{-0.8}$, (c) $f_{-0.9}$, and (d) $f_{-1.0}$. ($D_1 = 5.5$ and $d = 18$, $\alpha_{\text{max}} = \pi/4$).

histogram as $N_o(n)$, and its distance value $f$ as $f_o(n)$ (see Figure 4).

III. RESULTS AND DISCUSSION

According to the RT analysis of Qiu et al. [18, 19], the focal length $f_o$ can be expressed as

$$f_o = \frac{d}{|n|} - D_1.$$  \hspace{1cm} (3)

Figure 5 shows the $f_o$ and $N_o$ plots obtained for $\alpha_h \in [0^\circ, 45^\circ]$, $n \in [-0.7, -1.1]$, $d = 18$ cm, and $D_1 = 5.5$ cm, by using the MBI method. Figure 5(a) compares the results of the MBI method with the results of Equation (3); the results are comparable, showing good agreement.

In the case of $n = -1$, the angles $\alpha_h$ and $\alpha_c$ will be equal for all beams coming from a point source. In that case, the $f$ beam intersection distances become independent of the $\alpha_h$ and $\alpha_c$ angles and Equation (1) and Equation (3) can be simplified as $f = d - D_1$. As depicted in the RT analysis given in Figure 3(a) and demonstrated in the corresponding histogram in Figure 4(d), all beams intersect at the same $f$ distance. This effect is observed as a strong peak at $n = -1$ in the $N_o$ plot shown in Figure 5(b), and it implies that the most intensified beam intersection occurs at $n = -1$ and the quality of the focusing point...
FIG. 5: (a) Comparison of $f_o$ with Equation (3). (b) Corresponding $N_o$ plot and transmission rate (TR) inset plotting of the flat lens from the FDTD simulation, $(\alpha_h \in [0^\circ, 45^\circ], n \in [-0.7, -1.1], d = 18\text{ cm},$ and $D_1 = 5.5\text{ cm})$.

will be the best at $n = -1$.

The quality of the focusing point of a lens increases as the energy of the focalized waves intensifies at the narrowest region on the focal axis. In this sense, the $N_o$ values of the focusing points aid in assessing the quality of focusing.

The focal distance obtained by FDTD simulation of the sonic crystal flat lens for $n = -1$ is approximately 12 cm in Figure 2(c). This result is very consistent with the focal distance ($f_o$) obtained by the MBI method at $n = -1$, as shown in Figure 5(a). Transmission rates of flat lens calculated from the FDTD simulation were compared with the $N_o$ distribution on a focal axis in Figure 5(b). There is an apparent correlation between the transmission rates and $N_o$ distribution characteristics.

When $n \neq -1$, $N_o$ sharply decreases, since beam intersection points spread over the focal axis. This is a factor resulting in decreasing the focusing point intensity and thus the quality of focusing.

Figure 5(a) also reveals that the $f_o$ focal length alters depending on negative ERIs.
In previous works [19–21], the ERI of a sonic crystal was shown to be dependent on the frequency of the waves. This study confirms the finding that the focal distance of a flat lens can be adjusted by the frequency of the wave source [19]. However, the best focusing property will be obtained only in the frequency where the flat lens exhibits a negative ERI of \(-1\) \((n = -1)\).

In a recently study, Kasai et al. [22] discussed the incident-angle dependencies of the energy-transmission efficiency in detail. The wave frequency and structural parameters have an effect on the energy-transmission efficiency [22]. The beam intersection density analysis implies that the transmission efficiency for a point-type acoustic source also depends on the measurement location on the focal axis, due to the influence of spreading or intensification of the wave energy over the focal axis. Hence, the high transmission efficiency will be possible at the focusing point in the case of \(n = -1\), because there is a perfect convergence of wave beams on a single point of the focal axis.

**IV. CONCLUSIONS**

Histogram analysis of the beam intersection points in the RT analysis is an appropriate method for inspecting the focusing properties of flat lenses, because it considers the distribution of the beam intersection point density on the focal axis. Specifically, the peak density in the beam intersection distribution indicates wave energy focalization on this location, and gives an estimation of both the focal length and focusing quality of the negative refractive flat lens.

The proposed MBI method is in good agreement with the focal lengths calculated by using Equation (3). The FDTD simulation of flat lenses yields very consistent results with the MBI method. On the other hand, the MBI method provides additional information about the convergence of wave beams, which allows the assessment of the focusing point quality.

The MBI method advances the ray trace analysis of flat lenses, and it has the potential to contribute to negative refractive flat lens research in the field of metamaterial science.

**APPENDIX**

*Derivation of Equation (1):*

According to the geometrical analysis seen in Figure 6, the following equations can be written:

\[ w = D_1 \cdot \tan(\alpha_h) \quad \text{and} \quad v = f \cdot \tan(\alpha_h), \]

\[ d_w = w/ \tan(\alpha_c) \quad \text{and} \quad d_v = v/ \tan(\alpha_c). \]

The width of the crystal slab \((d)\) can be expressed as the sum of \(d_w\) and \(d_v\),

\[ d = d_w + d_v = (w + v)/ \tan(\alpha_c). \]
FIG. 6: Geometrical analysis for the derivation of Equation (1).

FIG. 7: A programming algorithm for the MBI method.
If the equations given for $w$ and $v$ are considered, $d$ can be reorganized as

$$d = \tan(\alpha_h)(D_1 + f)/\tan(\alpha_c).$$

Here, the $f$ distance is obtained as follows:

$$f = \frac{\tan \alpha_c d}{\tan \alpha_h} - D_1.$$

**Programming Algorithm for the computation of the MBI method:**

In the algorithm presented in Figure 7, the parameters $n_{\text{max}}$, $n_{\text{min}}$, and $\Delta n$ configure the upper bound, the lower bound, and the increment of the ERI, respectively. In addition, the parameters $\alpha_{\text{max}}$ and $\Delta \alpha$ are set at the beginning of the program to specify $\alpha_{\text{max}}$ in range of $[0, \alpha_{\text{max}}]$ with an increment of $\Delta \alpha$. The operator hist(.) denotes the histogram of the data set.

**References**