The Kepler Problem and the Isotropic Harmonic Oscillator

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We work out the full correspondence between the Kepler problem and the isotropic harmonic oscillator in Newtonian mechanics by means of a special transformation. We then apply this to get all the details of the Kepler problem from the simple solution of the isotropic harmonic oscillator. The usefulness of this transformation is also exemplified by finding the counterpart of the Laplace-Runge-Lenz vector for the isotropic harmonic oscillator.

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I. INTRODUCTION

The isotropic harmonic oscillator (linear force) and the Kepler problem (inverse square force) are two well known integrable cases in the central force problems. Their bound states have closed orbits for whatever initial conditions, and this is called the Bertrand theorem [1]. It is amazing that there exists a simple transformation relating the two problems [2, 3]. In this article we shall explore more elaborately this transformation. Since there is a simple solution for the isotropic harmonic oscillator, we work out how this transformation can yield us a solution for the Kepler Problem.

This article is organized as follows. In Section II, the full correspondence of the two problems is worked out in detail. Making use of the correspondence found (i.e., the transformation) we start with the simple solution of the isotropic harmonic oscillator to obtain the orbit of the Kepler problem in Section III. In Section IV, the time evolution of the trajectory of the Kepler problem is found by virtue of the additional transformation between the two time scales. In Section V, we study how to compute the counterpart of the Laplace-Runge-Lenz vector in the isotropic harmonic oscillator problem. Finally, in Section VI, we have a discussion and draw our conclusion.

II. THE EQUATIONS OF MOTION

It is convenient to employ complex numbers as the coordinates for central force problems, since the motion is essentially two dimensional in a plane orthogonal to the angular momentum vector, which is a conserved quantity. We let the complex number $Z = Z_1 + iZ_2$ denote the radius vector of the Kepler problem, and the complex number $w = w_1 + iw_2$ denote that for the isotropic harmonic oscillator. The equation of motion, i.e., the Newton equation for the Kepler problem, is then

$$m \frac{d^2 Z}{dt^2} = -k \frac{Z}{|Z|^3},$$ (1)
where \( k \) denotes the strength of the inverse square force and \( m \) is the mass of the particle in the force field. Analogously, the equation of motion for the harmonic oscillator is given as

\[
m\frac{d^2 w}{d\tau^2} = -m\omega^2 w, \tag{2}
\]

where \( m\omega^2 \) is the strength constant for the linear force. Notice that we employ two different time scales \( t \) and \( \tau \) separately for the two problems. It will be seen later that they are related by a nonlinear relation.

By virtue of the equations of motion it is easy to prove that the angular momentum for the separate problems is conserved, and we denote them as \( L \) and \( l \), respectively. Explicitly, their expressions are given as

\[
iL = \frac{m}{2} \left( \bar{Z} \frac{dZ}{dt} - \frac{d\bar{Z}}{dt} \bar{Z} \right), \tag{3}
\]

\[
il = \frac{m}{2} \left( \bar{w} \frac{dw}{d\tau} - \frac{d\bar{w}}{d\tau} \bar{w} \right). \tag{4}
\]

The simple transformation is just \( Z = w^2 \), but we need to keep the two angular momenta conserved and an easy way is to require \( L = l \). So additionally we must have

\[
Z \frac{dZ}{dt} = \bar{w} \frac{dw}{d\tau}. \tag{5}
\]

Together with the transformation \( Z = w^2 \) we should have

\[
2\bar{w}w \frac{d}{dt} = \frac{d}{d\tau}. \tag{6}
\]

Thus with this transformation, we have

\[
\frac{d^2 Z}{d\tau^2} = \frac{1}{2\bar{w}w} \frac{d}{d\tau} \left( \frac{1}{2\bar{w}w} \frac{d\bar{w}^2}{d\tau} \right) \tag{7}
\]

\[
= \frac{1}{2\bar{w}w} \left( \frac{1}{\bar{w}} \frac{d^2 \bar{w}}{d\tau^2} - \frac{1}{\bar{w}^2} \frac{d\bar{w}}{d\tau} \frac{d\bar{w}}{d\tau} \right) \tag{8}
\]

\[
= -\frac{w^2}{2(\bar{w}w)^3} \left( \frac{d\bar{w}}{d\tau} \frac{d\bar{w}}{d\tau} + \omega^2 \bar{w} \right) \tag{9}
\]

\[
= -\frac{E_h Z}{m|Z|^3}, \tag{10}
\]

where \( E_h \) is the total energy of the isotropic harmonic oscillator system. Thus we have the correspondence

\[
k = E_h. \tag{11}
\]
Furthermore we can consider the effect of the transformation on the energy conservation equation for the total energy $E_k$ of the Kepler problem.

$$E_k = \frac{m}{2} \frac{d\bar{\omega}}{d\tau} \frac{d\bar{w}}{d\tau} - \frac{k}{|\bar{Z}|}$$

$$= \frac{m\bar{\omega}w}{2(\bar{\omega}w)^2} \frac{d\bar{\omega}}{d\tau} \frac{dw}{d\tau} - \frac{k}{\bar{\omega}w}.$$  

After a simple algebraic manipulation we get

$$\frac{m}{2} \frac{d\bar{\omega}}{d\tau} \frac{dw}{d\tau} - E_k \bar{\omega} w = E_h.$$  

This yields another correspondence

$$|E_k| = \frac{1}{2} m\omega^2,$$  

where we only consider the bound states of the Kepler problem here. We believe that analytical continuation is possible for the scattering states.

### III. THE ORBIT EQUATIONS

It is relatively easy to work out the general solution $w(\tau)$ for the isotropic harmonic oscillator. With an appropriate choice of the axes and the initial conditions the trajectory $w(\tau)$ can be written as

$$w(\tau) = a \cos \omega \tau + ib \sin \omega \tau.$$  

The orbit is an ellipse on a plane, with semi-major axis of length $a$ and semi-minor axis of length $b$ and the force centre is at the centre of the ellipse.

Now we can make use of the transformation in the previous section to work out the orbit equation for the Kepler problem. Effecting the transformation, the radius vector $Z$ for the Kepler problem can be expressed as

$$Z(t) = a^2 \cos^2 \omega \tau - b^2 \sin^2 \omega \tau + 2iab \sin \omega \tau \cos \omega \tau$$

$$= \frac{a^2 - b^2}{2} + \frac{a^2 + b^2}{2} \cos 2\omega \tau + iab \sin 2\omega \tau.$$  

Of course $Z(t)$ should be expressed as depending on the time $t$, and we shall address such an issue in the next section. It is rather simple to get the orbit equation if we just eliminate the $\tau$ dependence. The orbit equation will be a displaced ellipse, with the length of the semi-major axis to be $A = \frac{a^2 + b^2}{2}$ and that of the minor axis to be $B = ab$. The eccentricity $\varepsilon$ equals $\frac{a^2 - b^2}{a^2 + b^2}$, and the force centre is at the focus. Let us note that $R_{max}$ and $R_{min}$ are separated by an angle of $\pi$ in the Kepler problem, while $r_{max}$ and $r_{min}$ are separated by an
angle of $\pi/2$ in the harmonic oscillator case. Indeed the angle is doubled, as is seen from
the transformation $Z = w^2$.

As usual we can express the two conserved quantities $E_k$ and $L$ in terms of the
geometrical quantities. The first useful identification is

$$E_k = -\frac{1}{2} m \omega^2$$

$$= -\frac{1}{2} m \omega^2 (a^2 + b^2)$$

$$= -\frac{k}{2A}.$$  \hfill (19)

$$= \frac{1}{2} m \omega^2 (a^2 + b^2)$$  \hfill (20)

Another identification for the angular momentum can be worked out as follows:

$$L^2 = m^2 \omega^2 a^2 b^2$$

$$= m^2 \omega^2 (a^2 + b^2)^2 \frac{a^2 b^2}{(a^2 + b^2)^2}$$

$$= mkA \frac{B^2}{A^2}.$$  \hfill (21)

This yields the relation

$$\varepsilon^2 = 1 + \frac{2E_k L^2}{mk^2}.$$  \hfill (22)

To conclude this section we would like to write the orbit equation in terms of the
polar coordinates $(R, \Theta)$. Indeed, we have

$$R = r^2 = a^2 \cos^2 \omega \tau + b^2 \sin^2 \omega \tau$$

$$= \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos 2\omega \tau,$$  \hfill (23)

and

$$R \cos \Theta = \frac{a^2 - b^2}{2} + \frac{a^2 + b^2}{2} \cos 2\omega \tau.$$  \hfill (24)

Therefore, the orbit equation is just

$$\frac{(1 - \varepsilon^2) A}{R} = 1 - \varepsilon \cos \Theta.$$  \hfill (25)

IV. ECCENTRIC ANOMALY

Now it is time to examine the time evolution of the trajectory of the radius vector
$Z(t)$ of the Kepler problem. We want to compute $R(t)$, the length of the radius vector.
Indeed, we have the formula

$$R(t) = A + \varepsilon A \cos 2\omega \tau.$$  \hfill (30)
Hence we have to solve explicitly for the transformation of the two time scales, \( t \) and \( \tau \). We have to integrate
\[
2(A + \varepsilon A \cos 2\omega \tau) d\tau = dt,
\]
and the relation is just
\[
2\omega \tau + \varepsilon \sin 2\omega \tau = \frac{\omega t}{A};
\]
We have to solve for \( \tau(t) \). However, this is a nonlinear relation where it is impossible to invert the relation analytically. The way out is to find the expression \( R(t) \) in a circuitous manner. We define a variable \( \Psi(t) \) as
\[
R(t) - A = \varepsilon A \cos \Psi(t),
\]
where \( \Psi(t) = 2\omega \tau \), and the variable \( \Psi(t) \) has to satisfy the relation
\[
\Psi(t) + \varepsilon \sin \Psi(t) = \sqrt{\frac{k}{mA^3}}t.
\]
The angular variable \( \Psi \) is known as the eccentric anomaly and the angle \( \Theta \) is called the true anomaly. Such variables are well studied in the Kepler problem.

Now we turn to the problem of the period of the trajectory. As is well known the period for the isotropic harmonic oscillator is \( 2\pi/\omega \) and is independent of the initial conditions, and this is because of the linear dependence \( \omega \tau \). On the other hand the period for the Kepler problem can be worked out by requiring that the eccentric anomaly changes by \( 2\pi \). We obtain
\[
T = 2\pi \sqrt{\frac{mA^3}{k}}.
\]
This is just the Kepler third law, which states that the square of the period of the planet is proportional to the cube of the average distance \( A \).

V. LAPLACE-RUNGE-LENZ VECTOR

In the Kepler problem besides the two conserved dynamical variables, the total energy and the angular momentum vector, there is an additional conserved vector, known as the Laplace-Runge-Lenz vector. The existence of this additional conserved vector indicates that the Kepler problem has a higher symmetry of SO(4). This conserved quantity is crucial for the working out of the hydrogen spectrum in the quantum theory. The Coulomb problem has the same mathematical structure as the Kepler problem since they are in the same class of the inverse square central force. In this section we shall focus on this conserved vector.
In the usual approach, the Laplace-Runge-Lenz vector $\mathbf{M}$ is defined in three dimensional space as

$$\mathbf{M} = p \times \mathbf{L} - \frac{mk}{r} \mathbf{r}.$$  \hspace{1cm} (36)

The vector lies on the orbit plane, and we can represent the vector as a complex number on the complex plane, without specifying the direction of the angular momentum vector. The expression is

$$\mu = mL \frac{dZ}{dt} - mk \frac{Z}{|Z|}.$$ \hspace{1cm} (37)

This time derivative is seen to be zero since

$$\frac{d}{dt} \frac{Z}{|Z|} = iL \frac{Z}{m |Z|^3}.$$ \hspace{1cm} (38)

One of the consequence of this conservation law is that $(mL \frac{dZ}{dt} - \mu)$ lies on a circle. The Laplace-Runge-Lenz vector points in the perihelion direction with magnitude $mk\varepsilon$.

The counterpart vector $\mu'$ of the Laplace-Runge-Lenz vector for the isotropic harmonic oscillator can be easily obtained through the transformation and is given by the expression

$$\mu' = \frac{1}{\bar{w}} \left( ml \frac{dw}{d\tau} - mE_h \bar{w} \right).$$ \hspace{1cm} (39)

It is easily shown to be a constant of motion, but it cannot be written in a simple vector form because of the prefactor $1/\bar{w}$. The formula suggests that $w^2$, i.e., parabolic coordinates, is a more appropriate coordinate frame for the discussion of such a conserved quantity. Lastly in this section we would like to mention that the magnitude of this conserved vector is

$$|\mu'| = \frac{1}{2} m^2 \omega^2 (a^2 - b^2).$$ \hspace{1cm} (40)

Together with the conservation of total energy, this implies $\frac{1}{2} m \omega^2 a^2$ and $\frac{1}{2} m \omega^2 b^2$ are separately conserved. So the existence of a higher symmetry of SO(4) is also valid for the isotropic harmonic oscillator.

VI. CONCLUSION AND DISCUSSION

It was previously proposed that the equation of motion for the central force Kepler problem can be obtained from that of the isotropic harmonic oscillator problem by a simple transformation $Z = w^2$ on the complex plane. Besides, time is also transformed accordingly. In this article we follow up on this idea, working out the full correspondence. We argue...
that the time scale transformation is necessary so that the two problems have the same value of the angular momentum. We also work out the correspondence of the two energy equations. It is seen that the total energy and the coupling (i.e., the potential energy) are interchanged in these two problems for the transformation. This interchange is apparent if we look more closely at the virial theorem for the two problems. Also the calculation is especially easy using the complex plane as the plane of the orbit. So there is a complex structure embedded in these two problems.

We make use of the simple solution for the isotropic harmonic oscillator to obtain the orbit equation in detail for the Kepler problem. There is no time involved, so we are just employing the conformal transformation \( Z = w^2 \). We can also calculate the time evolution \( Z(t) \), provided that we take into account the nonlinear change of the time scales. We also make use of the transformation to discuss the counterpart of the Laplace-Runge-Lenz vector in the isotropic harmonic oscillator.

It will be interesting to extend this correspondence of the Kepler problem and the isotropic oscillator for Lagrangian Mechanics to Hamiltonian Mechanics accordingly. The swapping of the coupling term and the total energy term may hint that the traditional formulation may require some changes. This is left for further study.

We would like to see if this correspondence can be extended to the quantum case. In quantum theory there is no equation of motion. Rather, the Hamiltonian or the energy levels are the physical variables. At present we do not know whether an analogous correspondence exists or not. The following argument offers a good boding that such a correspondence should exist. Consider the adiabatic invariant of the one-dimensional harmonic oscillator given by \( E_h/\omega \) [4]. It should be quantized. The equation of the isotropic harmonic oscillator is separable in the Cartesian coordinates. We can get the quantum formula \( E_h/\omega = n\hbar \), where \( n \) is an integer. Using the correspondence we have found we can easily derive the Bohr formula \( E_k = -\frac{mk^2}{2\hbar^2} \frac{1}{n^2} \). Of course this argument seems rather heuristic and ad hoc. The full symmetry of the Coulomb potential in the quantum case had previously been worked out in [5, 6]. So further study along this line may give us the answer rigorously.

References