Spectrum of a Hyperbolic Potential via SUSYQM within the Semi-Relativistic Formalism

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We apply the powerful technique of supersymmetric quantum mechanics to obtain approximate analytical solutions for the two-particle-spinless Salpeter equation, and thereby obtain both the eigenvalues and eigenenergies of the system for arbitrary quantum numbers. The numerical results are also included, which indicates an acceptable agreement between the two approaches.

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I. INTRODUCTION

The intermediate semi-relativistic spinless Salpeter equation (SSE) [1–5] is as attractive as the relativistic and nonrelativistic wave equations of quantum mechanics, as we frequently face phenomena with energies in this regime. On the other hand, two-body physical systems are present in a variety of physical fields. Because of these motivations, we intend to work on the SSE within the present study. The latter is an acceptable approximation to the Bethe-Salpeter equation, which describes the bound states within relativistic field theory. As can be inferred from the name of the equation, the spin degrees of freedom are dropped within the SSE equation. The treatment of the SSE is a difficult task because of its nonlocal nature. As a result, many authors have tried various approximate techniques to deal with the problem, a list of which can be found in the references [6–13].

Here, we use supersymmetric quantum mechanics (SUSYQM) to deal with a potential of hyperbolic form. Hyperbolic potentials, in many cases, are superior to their famous partners such a Coulomb or polynomial terms [14–22], and some authors have investigated them for either relativistic or nonrelativistic wave equations of mathematical physics. Nikiforov-Uvarov (NU) [23–25], SUSYQM [26], exact quantization [27], and series expansion techniques [28–30] have been used to deal with such problems.

Our work is organized as follows. We first review the two-body SSE as briefly as possible. Next, we apply a proper approximation to the centrifugal term. In the last step, we see that the problem is simply solved via the powerful quantum mechanical technique of SUSY.

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II. THE TWO-BODY-HAMILTONIAN

The SS equation for two particles which interact via a spherically symmetric potential in the center of mass system has the form \[28–35\]
\[
\sum_{i=1,2} \sqrt{-\Delta + m_i^2 + V(r)} - M \chi(\vec{r}) = 0, \quad \Delta = \nabla^2,
\]
where \(\chi(\vec{r}) = Y_{lm}^\text{m}(\theta, \phi)U_{nl}(r)\). In the case of heavy interacting particles, we may use the approximation \[36–40\]
\[
\sum_{i=1,2} \sqrt{-\Delta + m_i^2} = m_1 + m_2 - \frac{\Delta}{2\mu} - \frac{\Delta^2}{8\eta^3} - \cdots,
\]
with
\[
\mu = \frac{m_1m_2}{m_1 + m_2},
\]
\[
\eta = \mu \left(\frac{m_1m_2}{m_1m_2 - 3\mu^2}\right)^{1/3}.
\]

After some transformation, we arrive at \[36–40\]
\[
\left[ -\hbar^2 \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + W_{nl}(r) - \frac{W_{nl}^2(r)}{2\tilde{m}} \right] \psi_{nl}(r) = 0,
\]
where
\[
W_{nl}(r) = V(r) - E_{nl},
\]
\[
\tilde{m} = \eta^3/\mu^2 = (m_1m_2\mu)/(m_1m_2 - 3\mu^2).
\]

Here, we consider the potential in the form
\[
V(r) = V_0(1 - \coth(\alpha r)),
\]
where \(V_0\) and \(\alpha\) are constant coefficients. Before proceeding further, we make use of the approximation \[26–30\]
\[
\frac{1}{r^2} \approx \frac{\alpha^2}{\sinh^2(\alpha r)}.
\]
Substitution of (6) and (7) into (4) gives
\[
\left[ -\frac{d^2}{dr^2} + V_1 \text{csch}^2(\alpha r) - V_2 \coth(\alpha r) \right] \psi_{n,l}(r) = C \psi_{n,l}(r),
\]
where
\[
V_1 = \frac{W_{nl}^2}{2\tilde{m}}, \quad V_2 = \frac{W_{nl}^2}{2\tilde{m}}.
\]
where
\[ C = \frac{\mu}{\hbar^2 m} (E_{n,l} - V_0)^2 - \frac{2\mu}{\hbar^2} V_0 + \frac{2\mu}{\hbar^2} E_{n,l} + \frac{\mu V_0^2}{\hbar^2 m}, \]
\[ V_1 = \alpha^2 l(l + 1) - \frac{\mu V_0^2}{\hbar^2 m}, \]
\[ V_2 = \frac{\mu}{\hbar^2} \left( 2V_0 + \frac{2V_0 E_{n,l}}{m} - \frac{2V_0^2}{m} \right). \]

Eq. (8) can be written as
\[ -\frac{d^2 \psi_{n,l}(r)}{dr^2} + V_{\text{eff}}(r) \psi_{n,l}(r) = \tilde{E}_{n,l} \psi_{n,l}(r), \]
where
\[ V_{\text{eff}}(r) = V_1 \text{csch}^2(\alpha r) - V_2 \coth(\alpha r), \]
\[ \tilde{E}_{n,l} = C. \]

Bearing in mind Eq. (A-1), we search for a solution of the Riccati equation
\[ \phi^2 - \phi' = V_{\text{eff}} - \tilde{E}_{0,l}, \]
which is
\[ \phi(r) = A + B \coth(\alpha r). \]

From a glance at Eq. (11), the coefficients are found from
\[ A^2 + B^2 + B^2 \text{csch}^2(\alpha r) + 2AB \coth(\alpha r) + \alpha B \text{csch}^2(\alpha r) \]
\[ = +V_1 \text{csch}^2(\alpha r) - V_2 \coth(\alpha r) - \tilde{E}_{0,l}, \]
as
\[ B = -\alpha \pm \sqrt{\alpha^2 + 4V_1}, \]
\[ A = \frac{V_2}{2B}. \]

Therefore, our partner potentials are
\[ V_{\text{eff}+}(r) = \phi^2 + \frac{d\phi}{dr} = \text{csch}^2(\alpha r)[B^2 - \alpha B] + \frac{V_2^2}{4B^2} + B^2 - V_2 \coth(\alpha r), \]
\[ V_{\text{eff}-}(r) = \phi^2 - \frac{d\phi}{dr} = \text{csch}^2(\alpha r)[B^2 + \alpha B] + \frac{V_2^2}{4B^2} + B^2 - V_2 \coth(\alpha r), \]
which are shape invariant via a mapping of $B \rightarrow B - \alpha$. Thus, from Eq. (A-2) and its consequences we can get the energy:

\[
R(a_1) = \left( \frac{V_2^2}{4a_0^2} + a_0^2 \right) - \left( \frac{V_2^2}{4a_1^2} + a_1^2 \right),
\]

\[
R(a_2) = \left( \frac{V_2^2}{4a_1^2} + a_1^2 \right) - \left( \frac{V_2^2}{4a_2^2} + a_2^2 \right),
\]

\[
R(a_3) = \left( \frac{V_2^2}{4a_2^2} + a_2^2 \right) - \left( \frac{V_2^2}{4a_3^2} + a_3^2 \right),
\]

\[
\vdots
\]

\[
R(a_n) = \left( \frac{V_2^2}{4a_{n-1}^2} + a_{n-1}^2 \right) - \left( \frac{V_2^2}{4a_n^2} + a_n^2 \right),
\]

\[
\tilde{E}_{n,l} = \sum_{k=1}^{n} R(a_k) = \left( \frac{V_2^2}{4a_0^2} + a_0^2 \right) - \left( \frac{V_2^2}{4a_n^2} + a_n^2 \right),
\]

where $n = 0, 1, 2, \ldots$ and

\[
a_n = a_0 - n\alpha,
\]

\[
a_0 = B.
\]

From Eqs. (11) and (18) the eigenvalues are

\[
\tilde{E}_{n,l} = \tilde{E}_{n,l} - \tilde{E}_{0,l} = -(\frac{V_2^2}{4a_n^2} + a_n^2) = -(\frac{V_2^2}{4(B - \alpha)^2} + (B - n\alpha)^2),
\]

or, from Eq. (9),

\[
\frac{\mu}{\hbar^2} (E_{n,l} - V_0)^2 - \frac{2\mu}{\hbar^2} V_0 + \frac{2\mu V_2^2}{\hbar^2} E_{n,l} + \mu V_0^2 \frac{1}{\hbar^2} = -(\frac{V_2^2}{4(B - \alpha)^2} + (B - n\alpha)^2),
\]

which can be more explicitly expressed as

\[
E_{n,l} = \frac{1}{2(\frac{\mu}{\hbar^2} + \frac{V_0}{\hbar^2} \frac{1}{m})} \left[ -(2\mu \frac{V_0}{\hbar^2} - \frac{2\mu V_0}{m} + \frac{2\mu^2 V_2^2 m}{a_n^2 m} \left( \frac{1}{\hbar^2} - \frac{V_0}{m} \right)) \right.
\]

\[
\pm (2\mu \frac{V_0}{\hbar^2} - \frac{2\mu V_0}{m} + \frac{2\mu^2 V_2^2 m}{a_n^2 m} \left( \frac{1}{\hbar^2} - \frac{V_0}{m} \right))^2
\]

\[
-4(\frac{\mu}{m} + \frac{V_0}{a_n^2 m^2}) (a_n^2 - \frac{2\mu V_0}{m} \frac{1}{\hbar^2} + \frac{2\mu V_0^2}{m} + \frac{\mu^2 V_2^2}{a_n^2 m} (1 - \frac{V_0}{m} \frac{1}{\hbar^2})) \frac{1}{2}
\]

which fully determines the spectrum of the system. We have reported the obtained analytical energy spectrum in Table I, where a comparison with the numerical solution is included. We see that the results are acceptable.
TABLE I: A comparison of the obtained analytical energy spectrum with the numerical solution for \( \tilde{m} = 2, \mu = \frac{1}{2}, \hbar = 1, V_0 = 0.0099, \alpha = 0.01. \)

<table>
<thead>
<tr>
<th>( n, l )</th>
<th>( E_{n,l} ) (calculation)</th>
<th>( E_{n,l} ) (numerical)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0, 0 )</td>
<td>(-0.479469)</td>
<td>(-0.479459)</td>
</tr>
<tr>
<td>( 0, 1 )</td>
<td>(-0.0540605)</td>
<td>(-0.053996)</td>
</tr>
<tr>
<td>( 0, 2 )</td>
<td>(-0.018548)</td>
<td>(-0.018353)</td>
</tr>
<tr>
<td>( 0, 3 )</td>
<td>(-0.0070919)</td>
<td>(-0.006707)</td>
</tr>
<tr>
<td>( 1, 0 )</td>
<td>(-0.082859)</td>
<td>(-0.082859)</td>
</tr>
<tr>
<td>( 1, 1 )</td>
<td>(-0.019171)</td>
<td>(-0.019106)</td>
</tr>
<tr>
<td>( 1, 2 )</td>
<td>(-0.007191)</td>
<td>(-0.007001)</td>
</tr>
<tr>
<td>( 1, 3 )</td>
<td>(-0.002445)</td>
<td>(-0.002088)</td>
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<tr>
<td>( 2, 0 )</td>
<td>(-0.026854)</td>
<td>(-0.026850)</td>
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<td>( 2, 1 )</td>
<td>(-0.007435)</td>
<td>(-0.007373)</td>
</tr>
<tr>
<td>( 2, 2 )</td>
<td>(-0.002488)</td>
<td>(-0.002313)</td>
</tr>
</tbody>
</table>

III. CONCLUSION

We worked on the basis of the two-body spinless Salpeter equation, which possesses two notable attractive features: two-body formulation and a semi-relativistic nature. Because of the motivating results of hyperbolic potentials, we studied a potential of hyperbolic cotangent form. We first used a proper approximation, which brought the problem into a form solvable via the outstanding idea of SUSYQM. To test the reliability of the results, we next used a numerical program which yielded results close to the analytical approach. Our results are applicable for testing on spin-zero two-body systems whose energies lie in the intermediate range.

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APPENDIX A: SUPERSYMMETRY QUANTUM MECHANICS

In this appendix, we give a brief list of most essential formulas of SUSQM. These few lines are from [41–43]. Our first goal in SUSYQM is finding a solution of the Riccati
equation

\[ V_+ = \Phi^2 - \Phi' \] \hspace{1cm} (A1)

with \( V \) being the potential of the Schrödinger equation. If

\[ V_+ (a_0, x) = V_- (a_1, x) + R(a_1), \] \hspace{1cm} (A2)

where \( a_1 \) is a new set of parameters uniquely determined from the old set \( a_0 \) via the mapping \( F : a_0 \mapsto a_1 = F(a_0) \) and the residual term \( R(a_1) \) does not include \( x \), the partner potentials are shape invariant and the necessary information of the system is obtained via [41–43]

\[ E_n = \sum_{s=1}^{n} R(a_s), \] \hspace{1cm} (A3)

\[ \phi_n^- (a_0, x) = \prod_{s=0}^{n-1} \left( \frac{A^\dagger (a_s)}{|E_n - E_s|^{1/2}} \right) \phi_0^- (a_n, x), \] \hspace{1cm} (A4)

\[ \phi_0^- (a_n, x) = C \exp \left\{ - \int_0^x dz \Phi (a_n, z) \right\}, \] \hspace{1cm} (A5)

\[ A_s^\dagger = - \frac{\partial}{\partial x} + \Phi (a_s, x). \] \hspace{1cm} (A6)

References