One-Step Generation of Maximally Entangled States for Three Atoms Trapped in Separated Cavities

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We propose a scheme for implementing Greenberger-Horne-Zeilinger states for three atoms trapped in distant cavities connected by optical fibers. Under the large detuning condition, the three distant atoms are coupled via four-photon processes induced by the vacuum bosonic modes and a classical field, leading to a phase shift proportional to the square of the atomic population in a certain state. During the process, the atoms do not undergo any real transition, the cavity modes and the fiber mode are only virtually excited, and the three-atom entanglement is produced in one step.

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The superposition of product states for two or more particles in a Hilbert space gives rise to entanglement that cannot be explained in classical terms. An entangled state is not only an essential ingredient for the testing of quantum mechanics against local hidden variable theories [1, 2], but also a key resource for the implementation of quantum information processing tasks, such as quantum cryptography [3], teleportation [4], and computation [5]. In essence, the implementation of a quantum computational task corresponds to the preparation and manipulation of an entangled state for the qubits on the quantum register. In recent years, there has been much interest in the generation of entangled states in various physical systems. Of special interest are the multi-particle maximally entangled states, i.e., Greenberger-Horne-Zeilinger (GHZ) states [2]. Such states can be used to test quantum nonlocality without using the Bell inequalities. The cavity QED system with atoms interacting with quantized electromagnetic fields is almost an ideal candidate for implementing the tasks of quantum information processing, because the atoms are suitable for storing information and photons are suitable for transporting information. Three-particle entanglement has been demonstrated within a microwave cavity [6]. However, the fidelity needs to be significantly improved for the testing of quantum nonlocality and quantum information processing.

Entanglement and controlled-phase gates between two distant atoms have been proposed [7, 8]. The schemes are probabilistic, and the success probability depends upon the efficiency of the photodetectors. Schemes have been proposed for deterministically realizing quantum gates between two two-level atoms in distant optical cavities mediated by an optical fiber [9–11]. These schemes are based on the Rabi oscillation of the whole system composed of the two atoms, cavity modes, and the fiber mode. Methods based on adia-

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batic following of the dark states have also been suggested to engineer entanglement and quantum logic operations between distant atomic qubits [12–15]. An alternative scheme has been suggested for controlled-phase gates between two atoms trapped in two distant optical cavities via a virtual excitation and transmission of photons [16]. More recently, it has been shown that controlled coherent couplings for \( n \) identical atoms trapped in separated cavities connected by optical fibers can be realized via pairing the off-resonant Raman transitions [17]. In order to realize the \( XY \) coupling the scheme requires each atom to be driven by two classical fields and a periodic boundary condition to be satisfied (the \( n \)th cavity is connected to the first cavity).

In this paper we propose a scheme for the generation of GHZ states for three atoms trapped in distant optical cavities. Each atom is coupled to the respective cavity mode and driven by a classical field, and each pair of adjacent cavities are coupled via an optical fiber. Under the large detuning condition, the bosonic modes are coupled to the classical fields via the virtual excitation of the atoms. Through a suitable choice of the detunings and Rabi frequencies of the classical fields, the nonresonant coupling between the quantized bosonic modes and classical fields produces a phase shift proportional to the square of the atomic population in a certain state. The GHZ state is generated in a single step. In contrast with the scheme of Ref. [17], this scheme does not require the pairing of the off-resonant Raman transitions and only a single classical field is applied to each atom. Nor does it require a periodic boundary condition to be satisfied.

Suppose that three identical atoms are trapped in distant cavities, which are connected by two optical fibers, as shown in Fig. 1. In the short fiber limit, only one fiber mode essentially interacts with the cavity modes [9]. In this case the coupling between the cavity modes and fibers is given by the interaction Hamiltonian

\[
H_{c,f} = \sum_{j=1}^{2} \nu \left[ b_j \left( a_j^\dagger + a_{j+1}^\dagger \right) \right] + \text{H.c.},
\]

where \( b_j \) is the annihilation operator for the \( j \)th fiber mode, \( a_j^\dagger \) is the creation operator for the \( j \)th cavity mode, and \( \nu \) is the cavity-fiber coupling strength. The atomic level configuration is shown in Fig. 2, where \( |e\rangle \) represents one excited state and \( |g\rangle \) and \( |f\rangle \) represent two hyperfine ground states. The transition \( |e\rangle \leftrightarrow |g\rangle \) for the \( j \)th atom is coupled to the corresponding cavity mode with the coupling constant \( g \) and a classical laser field with the Rabi frequency \( \Omega_j \) and phase \( \varphi_j \). The detunings of the respective cavity mode and classical field are \( \Delta \) and \( \Delta - \delta \), respectively. During the interaction, the state \( |f\rangle \) is not affected, since the transition \( |e\rangle \leftrightarrow |f\rangle \) is far off-resonant with the fields. In the interaction picture, the Hamiltonian describing the atom-cavity interaction is

\[
H_{a,c} = \sum_{j=1}^{3} \left[ \left( g a_j e^{i\Delta t} + \Omega_j e^{i\varphi_j} e^{i(\Delta-\delta)t} \right) |e_j\rangle \langle g_j| + \text{H.c.} \right].
\]
Define
\[ c_1 = \frac{1}{\sqrt{3}} (a_1 - a_2 + a_3), \]
\[ c_2 = \frac{1}{2} (a_1 + b_1 - b_2 - a_3), \]
\[ c_3 = \frac{1}{2} (a_1 - b_1 + b_2 - a_3), \]
\[ c_4 = \frac{1}{\sqrt{12}} \left( a_1 + \sqrt{3} b_1 + 2a_2 + \sqrt{3} b_2 + a_3 \right), \]
\[ c_5 = \frac{1}{\sqrt{12}} \left( a_1 - \sqrt{3} b_1 + 2a_2 - \sqrt{3} b_2 + a_3 \right). \]
(3)

Then we can rewrite the whole Hamiltonian in the interaction picture as
\[ H = H_0 + H_i, \]
(4)
where
\[ H_0 = \nu c_2^\dagger c_2 - \nu c_3^\dagger c_3 + \sqrt{3} \nu c_4^\dagger c_4 - \sqrt{3} \nu c_5^\dagger c_5, \]
(5)
and
\[ H_i = \left\{ \frac{1}{2} g \left[ \frac{2\sqrt{3}}{3} c_1 + c_2 + c_3 + \sqrt{3} (c_4 + c_5) \right] e^{i\Delta t} + \Omega e^{i(\Delta - \delta)t} \right\} |e_1\rangle \langle g_1| \\
- \frac{\sqrt{3}}{3} \left\{ [c_1 - (c_4 + c_5)] e^{i\Delta t} + r \Omega e^{i(\Delta - \delta)t} \right\} |e_2\rangle \langle g_2| \\
+ \left\{ \frac{1}{2} g \left[ \frac{2\sqrt{3}}{3} c_1 - c_2 - c_3 + \sqrt{3} (c_4 + c_5) \right] e^{i\Delta t} + \Omega e^{i(\Delta - \delta)t} \right\} |e_3\rangle \langle g_3| + \text{H.c.} \]  
(6)

We here have set \( \Omega_1 = \Omega_3 = \Omega, \Omega_2 = r \Omega, \varphi_1 = \varphi_3 = 0, \) and \( \varphi_2 = \pi. \)

FIG. 1: The experimental setup. Three distant atoms are trapped in separate cavities connected by two optical fibers.

The time evolution of this system is governed by Schrödinger’s equation:
\[ i \frac{d|\psi(t)\rangle}{dt} = H |\psi(t)\rangle. \]
(7)
Performing the unitary transformation
\[ |\psi(t)\rangle = e^{-iH_0 t} |\psi'(t)\rangle, \]
(8)
FIG. 2: The atomic level configuration. The transition $|e⟩ \leftrightarrow |g⟩$ is coupled to the corresponding cavity mode and driven by a classical laser field with the Rabi frequency $\Omega$. The detunings of the cavity mode and classical field are $\Delta$ and $\Delta - \delta$, respectively.

we obtain

$$i \frac{d|\psi'(t)\rangle}{dt} = H'_i |\psi'(t)\rangle,$$ (9)

where

$$H'_i = \left\{ \frac{1}{2} g \left[ \frac{2\sqrt{3}}{3} e^{i\Delta t} c_1 + e^{i(\Delta - \nu)t} c_2 + e^{i(\Delta + \nu)t} c_3 + \frac{\sqrt{3}}{3} \left( e^{i(\Delta - \sqrt{3}\nu)t} c_4 + e^{i(\Delta + \sqrt{3}\nu)t} c_5 \right) + \Omega e^{i(\Delta - \delta)t} \right] |e_1⟩⟨g_1| \right. \right.$$ 

$$- \left. \frac{\sqrt{3}}{3} \left\{ g \left[ e^{i(\Delta t)} c_1 - e^{i(\Delta - \sqrt{3}\nu)t} c_4 + e^{i(\Delta + \sqrt{3}\nu)t} c_5 \right] + r\Omega e^{i(\Delta - \delta)t} \right\} |e_2⟩⟨g_2| \right.$$ 

$$+ \left. \left\{ \frac{1}{2} g \left[ \frac{2\sqrt{3}}{3} e^{i\Delta t} c_1 - e^{i(\Delta - \nu)t} c_2 - e^{i(\Delta + \nu)t} c_3 + \frac{\sqrt{3}}{3} \left( e^{i(\Delta - \sqrt{3}\nu)t} c_4 + e^{i(\Delta + \sqrt{3}\nu)t} c_5 \right) + \Omega e^{i(\Delta - \delta)t} \right] |e_3⟩⟨g_3| + \text{H.c.} \right\}. \quad (10) $$

Suppose that the atoms are initially populated in the ground states and the bosonic modes in the vacuum states. Under the condition $\Delta \gg \sqrt{3}\nu$, $\delta, g$ the atoms do not exchange energy with the fields and remain in the ground states. The bosonic modes can be coupled to the classical fields via the virtual excitation of the atoms. We can use the formula presented by James and Jerke [18] to derive the effective Hamiltonian describing the dynamics of the detuned quantum system. The effective coupling between the bosonic mode $c_1$ and the corresponding classical field due to their off-resonant interaction with atom 1 is described
by [18]
\[
\frac{1}{\Delta_1} \left[ \frac{\sqrt{3}}{3} g c_1 |e_1\rangle \langle g_1|, \Omega |g_1\rangle \langle e_1| \right] \text{e}^{i\delta t} + \text{H.c.}
\]
\[
= \frac{\sqrt{3} g \Omega}{3 \Delta_1} c_1 [e_1 - |g_1\rangle \langle g_1|] \text{e}^{i\delta t} + \text{H.c.},
\]
where
\[
\frac{1}{\Delta_1} = \frac{1}{2} \left( \frac{1}{\Delta} + \frac{1}{\Delta - \delta} \right).
\]
(11)

In the same way we can obtain the effective coupling between each bosonic mode and the corresponding classical field due to their off-resonant interaction with each atom. Since the atoms are initially in the ground states, the terms proportional to \(|e_j\rangle \langle e_j|\) have no effect on the state evolution and thus can be discarded. So the total effective Hamiltonian \(H_e\) is given by
\[
H_e = - \left[ \lambda_1 \text{e}^{i\delta t} c_1 + \lambda_2 \text{e}^{i(\delta - \nu)t} c_2 + \lambda_3 \text{e}^{i(\delta + \nu)t} c_2 + \lambda_4 \text{e}^{i(\delta - \sqrt{3}\nu)t} c_4 + \lambda_5 \text{e}^{i(\delta + \sqrt{3}\nu)t} c_5 \right] |g_1\rangle \langle g_1|
\]
\[
- r \left[ \lambda_1 \text{e}^{i\delta t} c_1 - 2 \lambda_4 \text{e}^{i(\delta - \sqrt{3}\nu)t} c_4 - 2 \lambda_5 \text{e}^{i(\delta + \sqrt{3}\nu)t} c_5 \right] |g_2\rangle \langle g_2|
\]
\[
- \left[ \lambda_1 \text{e}^{i\delta t} c_1 - \lambda_2 \text{e}^{i(\delta - \nu)t} c_2 - \lambda_3 \text{e}^{i(\delta + \nu)t} c_2 + \lambda_4 \text{e}^{i(\delta - \sqrt{3}\nu)t} c_4 + \lambda_5 \text{e}^{i(\delta + \sqrt{3}\nu)t} c_5 \right] |g_3\rangle \langle g_3|
\]
\[
+ \text{H.c.} - \left( \eta + \varepsilon_1 c_1^\dagger c_1 + \varepsilon_2 c_2^\dagger c_2 + \varepsilon_3 c_3^\dagger c_3 + \varepsilon_4 c_4^\dagger c_4 + \varepsilon_5 c_5^\dagger c_5 \right) (|g_1\rangle \langle g_1| + |g_3\rangle \langle g_3|)
\]
\[
- \left( r^2 \eta + \varepsilon_1 c_1^\dagger c_1 + 4 \varepsilon_4 c_4^\dagger c_4 + 4 \varepsilon_5 c_5^\dagger c_5 \right) |g_2\rangle \langle g_2|,
\]
(13)

where
\[
\lambda_1 = \frac{\sqrt{3} g \Omega}{6} \left( \frac{1}{\Delta} + \frac{1}{\Delta - \delta} \right), \quad \lambda_2 = \frac{g \Omega}{4} \left( \frac{1}{\Delta - \nu} + \frac{1}{\Delta + \delta} \right),
\]
\[
\lambda_3 = \frac{g \Omega}{4} \left( \frac{1}{\Delta - \nu} + \frac{1}{\Delta - \delta} \right), \quad \lambda_4 = \frac{\sqrt{3} g \Omega}{12} \left( \frac{1}{\Delta - \sqrt{3}\nu} + \frac{1}{\Delta - \delta} \right),
\]
\[
\lambda_5 = \frac{\sqrt{3} g \Omega}{12} \left( \frac{1}{\Delta + \sqrt{3}\nu} + \frac{1}{\Delta - \delta} \right),
\]
\[
\eta = \frac{\Omega^2}{\Delta - \delta}, \quad \varepsilon_1 = \frac{g^2}{3 \Delta},
\]
\[
\varepsilon_2 = \frac{g^2}{4(\Delta - \nu)}, \quad \varepsilon_3 = \frac{g^2}{4(\Delta + \nu)},
\]
\[
\varepsilon_4 = \frac{g^2}{12(\Delta - \sqrt{3}\nu)}, \quad \varepsilon_4 = \frac{g^2}{12(\Delta + \sqrt{3}\nu)}.
\]
(14)
Under the conditions $\delta, \nu, |\delta \pm \nu|, |\delta \pm \sqrt{3}\nu| \gg \lambda_j$ the bosonic modes can not exchange energy with the classical fields and remain in the vacuum states. The nonresonant coupling between the bosonic modes and the classical fields leads to the coupling between the atoms. Using the effective Hamiltonian theory again, we can obtain the further effective Hamiltonian

$$H'_e = \mu_1 (|g_1\rangle \langle g_1| + |g_3\rangle \langle g_3|) + \mu_2 |g_2\rangle \langle g_2| + \xi_1 (|g_1\rangle \langle g_1| \otimes |g_2\rangle \langle g_2| + |g_1\rangle \langle g_1| \otimes |g_3\rangle \langle g_3|) + \xi_2 |g_1\rangle \langle g_1| \otimes |g_3\rangle \langle g_3|$$

$$- \left(\eta + \varepsilon_1 c_1^\dagger c_1 + \varepsilon_2 c_2^\dagger c_2 + \varepsilon_3 c_3^\dagger c_3 + \varepsilon_4 c_4^\dagger c_4 + \varepsilon_5 c_5^\dagger c_5\right) (|g_1\rangle \langle g_1| + |g_3\rangle \langle g_3|)$$

$$- \left(r^2 \eta + \varepsilon_1 c_1^\dagger c_1 + 4\varepsilon_4 c_4^\dagger c_4 + 4\varepsilon_5 c_5^\dagger c_5\right) |g_2\rangle \langle g_2|,$$

where

$$\mu_1 = \frac{\lambda_1^2}{\delta} + \frac{\lambda_2^2}{\delta + \nu} + \frac{\lambda_3^2}{\delta - \nu} + \frac{\lambda_4^2}{\delta + \sqrt{3}\nu} + \frac{\lambda_5^2}{\delta - \sqrt{3}\nu},$$

$$\mu_2 = r^2 \left(\frac{\lambda_1^2}{\delta} + \frac{4\lambda_2^2}{\delta + \sqrt{3}\nu} + \frac{4\lambda_5^2}{\delta - \sqrt{3}\nu}\right),$$

$$\xi_1 = 2r \left(\frac{\lambda_1^2}{\delta} - \frac{2\lambda_3^2}{\delta + \sqrt{3}\nu} - \frac{2\lambda_5^2}{\delta - \sqrt{3}\nu}\right),$$

$$\xi_2 = 2 \left(\frac{\lambda_1^2}{\delta} - \frac{\lambda_2^2}{\delta + \nu} - \frac{\lambda_3^2}{\delta - \nu} + \frac{\lambda_4^2}{\delta + \sqrt{3}\nu} + \frac{\lambda_5^2}{\delta - \sqrt{3}\nu}\right).$$

The Hamiltonian describes a four-photon process which is induced by the virtual excitation of the atoms and bosonic modes. The total photon number of the bosonic modes is conserved during the interaction. Suppose that all the two cavity modes and fiber modes are initially in the vacuum state. Then they would remain in the vacuum state during the evolution, which means $c_1^\dagger c_1 = c_2^\dagger c_2 = c_3^\dagger c_3 = c_4^\dagger c_4 = c_5^\dagger c_5 = 0$. In this case the effective Hamiltonian $H'_e$ becomes

$$H'_e = (\mu_1 - \eta) (|g_1\rangle \langle g_1| + |g_3\rangle \langle g_3|) + (\mu_2 - r^2 \eta) |g_2\rangle \langle g_2|$$

$$+ \xi_1 (|g_1\rangle \langle g_1| \otimes |g_2\rangle \langle g_2| + |g_1\rangle \langle g_1| \otimes |g_3\rangle \langle g_3|)$$

$$+ \xi_2 |g_1\rangle \langle g_1| \otimes |g_3\rangle \langle g_3|.$$ (17)

Choose the parameter $r$ appropriately, so that $\xi_1 = \xi_2 = \xi$. Then we have

$$H'_e = H_{e,0} + H_{e,i},$$ (18)

where

$$H_{e,0} = \sum_{j=1}^{3} \gamma_j |g_j\rangle \langle g_j|,$$

$$H_{e,i} = \frac{\xi}{2} \left(\sum_{j=1}^{3} |g_j\rangle \langle g_j|\right)^2.$$ (19)
and
\[
\gamma_1 = \gamma_3 = \mu_1 - \eta - \frac{\xi^2}{4}, \\
\gamma_2 = \mu_2 - r^2 \eta - \frac{\xi^2}{4}. 
\]
(20)

The evolution operator of the system is \( e^{-iH_{\text{e.o.t}}t} e^{-iH_{\text{e.o.t}}t} \).

Assume that each atom is initially in the state \( \frac{1}{\sqrt{2}} (|g_j\rangle + |f_j\rangle) \). Then the state for the atomic system can be written as [19]
\[
|\psi(0)\rangle = \frac{1}{\sqrt{2^3}} \sum_{k=0}^{3} \phi_k, 
\]
(21)

where
\[
|\phi_0\rangle = |f_1 f_2 f_3\rangle, \\
|\phi_1\rangle = (|g_1 f_2 f_3\rangle + |f_1 g_2 f_3\rangle + |f_1 f_2 g_3\rangle), \\
|\phi_2\rangle = (|g_1 g_2 f_3\rangle + |g_1 f_2 g_3\rangle + |f_1 g_2 g_3\rangle), \\
|\phi_3\rangle = |g_1 g_2 g_3\rangle. 
\]
(22)

After an interaction time \( \tau \) we have
\[
|\psi(\tau)\rangle = \frac{1}{\sqrt{2^3}} e^{-iH_{\text{e.o.t}}t} \sum_{k=0}^{3} e^{-ik^2 \xi \tau / 2} |\phi_k\rangle. 
\]
(23)

With the choice \( \xi \tau = \pi \), we obtain the GHZ state
\[
|\psi(\tau)\rangle = \frac{1}{\sqrt{2^3}} e^{-iH_{\text{e.o.t}}t} \sum_{k=0}^{3} (-i)^k |\phi_k\rangle \\
= \frac{1}{4} e^{-iH_{\text{e.o.t}}t} \sum_{k=0}^{3} e^{i\pi / 4 (-1)^k + e^{-i\pi / 4}} |\phi_k\rangle \\
= \frac{1}{\sqrt{2}} e^{i\pi / 4} (|-1 - 2 - 3\rangle - i |+1 + 2 + 3\rangle), 
\]
(24)

where
\[
|+j\rangle = \frac{1}{\sqrt{2}} (|f_j\rangle + e^{-i\gamma_j t} |g_j\rangle), \\
|-j\rangle = \frac{1}{\sqrt{2}} (|f_j\rangle - e^{-i\gamma_j t} |g_j\rangle). 
\]
(25)

During the operation, all the the atoms, cavity modes, and fiber modes are not excited, and the system evolves in the decoherence-free subspace.
We briefly address the experimental feasibility of the proposed scheme. Set $\Omega = g$, $r = 1.5027$, $\Delta = 20g$, $\delta = g$, $\nu = \sqrt{3}g$, and $\Gamma = \kappa = 0.01g$, where $\Gamma$ and $\kappa$ are the decay rates for the atomic excited state and the bosonic modes, respectively. Then we have $\lambda_1 \simeq 0.0296g$, $\lambda_2 \simeq 0.02465g$, $\lambda_3 \simeq 0.02685g$, $\lambda_4 \simeq 0.0139g$, $\lambda_5 \simeq 0.016g$, and $\xi \simeq 3.12 \times 10^{-3}g$. The time needed to generate the three-atom GHZ state is $t = 320.5128\pi/g$. During the procedure all the atoms and bosonic modes are only virtually excited. The effective decoherence rates due to the atomic spontaneous emission and the decay of the bosonic modes are $\Gamma' = \Gamma\Omega^2/\Delta^2 = 2.5 \times 10^{-3}g$ and $\kappa' = \left[\lambda_1^2/\delta^2 + \lambda_2^2/(\delta+\nu)^2 + \lambda_3^2/(\delta-\nu)^2 + \lambda_4^2/(\delta+\sqrt{3}\nu)^2 + \lambda_5^2/(\delta-\sqrt{3}\nu)^2\right]\kappa \simeq 0.825 \times 10^{-4}g$, respectively. The infidelity induced by the decoherence is on the order of $(\Gamma' + \kappa')t \simeq 4.9 \times 10^{-2}$. The parameters reported in the microsphere cavity QED experiment are $g \simeq 2\pi \times 20$ MHz, $\gamma \simeq 2\pi \times 2.6$ MHz, and $\kappa \simeq 2\pi \times 7$ MHz [20]. The corresponding cooperativity factor $C = g^2/2\gamma\kappa$ is too low for the implementation of a high-fidelity GHZ state, however the cooperativity of a value about $10^5$ is predicted to be available [21], with which a fidelity of 98% can be obtained. The near-perfect fiber-cavity coupling with an efficiency larger than 99.9% has been achieved using fiber-taper coupling to high-Q silica microspheres [22]. The fiber decay rate at 852 nm wavelength is about $1.52 \times 10^5$ Hz [23], much smaller than the available cavity decay rate. Therefore, it is conservative to set the decay rates of all the bosonic modes to be equal to the cavity decay rate.

In conclusion, we have proposed a scheme for generating GHZ states for three atoms trapped in separate cavities connected by optical fibers. Our scheme is based on the long-range atom-atom coupling induced by the virtual excitations of the atoms, cavity modes, and fiber modes. Through suitable choice of the parameters of the classical fields, this coupling produces a phase shift proportional to the square of the atomic population in a certain state, driving the atomic system from a product state to the maximally entangled state. In the scheme all the atoms and bosonic modes are not excited, and thus the scheme is insensitive to the atomic spontaneous emission, cavity decay, and fiber loss.

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References