Effects of Time Delay on Resonant Activation over a Fluctuating Barrier Driven by Correlated Noises

Fei Long* and Dongcheng Mei

Department of Physics, Yunnan University, Kunming 650091, China

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The time delay effects on the mean first passage time (MFPT) of a particle over a fluctuating potential barrier in the presence of correlated noises are investigated. It is shown that the MFPT displays the resonant activation (RA) phenomenon which can be enhanced by time delay feedbacks. In the non-delayed case, with increasing noise correlation intensity, RA is strengthened. However, under the synergistic reaction of time delay and noise correlation, there is a critical behavior which distinguishes resonant activation from resonant restraint at the corresponding critical value $\lambda = 0.02$, which is a so-called critical phenomenon.

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I. INTRODUCTION

Ever since the seminal achievements of Kramers and Arrhenius, the conventional problem of escape over a fluctuating barrier continues to attract ever growing attention. The concept of resonant activation (RA) was firstly put forward by Doering and Gadoua [1]; it shows that the mean first passage time (MFPT) of a particle driven by additive noises over a fluctuating potential barrier exhibits a minimum as a function of the potential barrier flipping rate. Hänggi et al. studied colored noise-assisted escape over fluctuating barriers, for it involves several relevant time-scales [2]; they also investigated the rates of activated processes with fluctuating barriers and gave a simple physical interpretation of RA [3]. Bag and Hu et al. investigated the MFPT for the particle to escape from an unstable limit cycle and the RA phenomenon [4, 5]; they also studied the effect of time delay on the onset of synchronization of the stochastic Kuramoto model [6]. Spagnolo et al. have a long-term investigation on the MFPT versus different parameters and presented an experimental study detecting RA [7]. They made further effort on the relationship between the RA phenomenon and the noise-enhanced stability (NES) effect and found the co-occurrence of RA and NES in a model of cancer growth [8]. Reimann sufficiently discussed the MFPT as a function of correlation time and reported a new type of RA [13, 14]. Li developed a series of excellent studies, which indicated that the RA phenomenon also occurs when the MFPT has a minimum as a function of the transition rate of the dichotomous noise [15–18]. A variety of types of noises were investigated by Li and co-workers to reveal the essence of RA. Dichotomous noise and Gaussian white noise could induce RA in a fluctuating barrier.
system; there are three or more RAs when the fluctuating potential barrier is driven by two or more dichotomous noises [15]. The effect of additive and multiplicative noises with correlations between them on RA is that the additive and multiplicative noises can weaken the RA, however the correlation between them can enhance it [16]. If the noise is three-state Markovian, the RA phenomenon will exist no matter what the potential is [17, 18]. Refs. [17–19] gives some kinetic approaches to the problem of diffusion over a fluctuating potential [19–21].

All the above mentioned works revealed a RA of the MFPT as a function of some specific parameters in non-delayed systems. However, due to the finite transmission times of the signals or other key quantities, such as matter, information, and energy, time delay is ubiquitous in nature [22, 23]. One unavoidable problem is how is the situation if we introduce time delay into a traditional but concise system. Nie and Mei reported that time delay changes the stationary probability distribution of a bistable system [9], and then they have shown that time delay induces the densities of the two species to periodically oscillate synchronously in the Lotka-Volterra model [10]. Time delay suppresses the stochastic resonance in the signal-to-noise ratio as a function of the noise intensities reported by Du and Mei [11]. Jia investigated the effects of time delay on the transient properties of a metastable system and reported an interesting critical phenomenon [12]. Under the synergistic reaction of time delay and noise correlation, the system presents different steady properties with regular situations. There are some interesting phenomena such as critical behavior, resonant activation, and resonant restraint worthy of our attention. In this paper, we shall investigate the time delay effects on the mean first passage time over a fluctuating potential barrier in the presence of an additive noise and a multiplicative noise. For simplicity and convenience, we take the fluctuating potential barrier as the piecewise linear one depicted in Fig. 1. In our study, we will use the definition of the MFPT to obtain by numerical integration an explicit expression for the MFPT in the case of time delay. According to the expression, we can give figures to illuminate the relationship between the MFPT and the flipping rate $\gamma$. Then a conclusion and discussion summarize the paper.

II. THE MODEL AND ITS MASTER EQUATION

Consider a one-dimensional model of an overdamped particle driven by the simultaneous action of an additive noise and a multiplicative noise. The dimensionless form of its Langevin equation reads

$$\frac{dx}{dt} = -\frac{\partial}{\partial x} U(x, t) + x(t)\xi(t) + \eta(t),$$

where $\xi(t)$ and $\eta(t)$ are Gaussian white noises with zero mean and have the following statistical properties:

$$\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t'),$$

$$\langle \eta(t)\eta(t') \rangle = 2Q\delta(t - t'),$$

$$\langle \xi(t)\eta(t') \rangle = 2\langle \xi(t)\xi(t') \rangle + 2\langle \eta(t)\eta(t') \rangle.$$
FIG. 1: Setup for the problem. The center height of the triangular potential barrier fluctuates between $E + \Delta E$ and $E - \Delta E$ at the rate $\gamma$.

$$\langle \xi(t) \eta(t') \rangle = \langle \eta(t) \xi(t') \rangle = 2\lambda \sqrt{DQ} \delta(t - t').$$  \hspace{1cm} (4)

Here $D$ and $Q$ are the multiplicative and additive noise intensities, respectively. $\lambda$ denotes the cross-correlated intensity between $\xi(t)$ and $\eta(t)$. The potential $U(x)$ is piecewise linear and fluctuating; it is plotted in Fig. 1 and is determined by the expression

$$U(x) = \begin{cases} 2Ex & \text{when } 0 \leq x \leq 1/2, \\ -2Ex + 2E & \text{when } 1/2 \leq x \leq 1. \end{cases}$$  \hspace{1cm} (5)

Taking the time delay effect into consideration, according to the probability density approach in Ref. [24] for linear time-delayed feedback, the effective potential is given by the following function [23, 24]:

$$U(x, \tau) = \begin{cases} 2(1 + \tau)Ex & \text{when } 0 \leq x \leq 1/2, \\ -2(1 + \tau)Ex + 2(1 + \tau)E & \text{when } 1/2 \leq x \leq 1, \end{cases}$$  \hspace{1cm} (6)

where $\tau$ is the delay time and we substituted $x, \tau$ for $x(t - \tau)$.

The joint probability distribution $P_\pm(x, t; x, t - \tau)$ for the particle at $x$ is given by the following the Fokker-Planck Equation (FPE) [25–28]:

$$\frac{\partial}{\partial t} \left( \begin{array}{c} P_+(x, t; x, t - \tau) \\ P_-(x, t; x, t - \tau) \end{array} \right) = \left( \begin{array}{cc} G^+ & \gamma \\ G^- & -\gamma \end{array} \right) \left( \begin{array}{c} P_+(x, t; x, t - \tau) \\ P_-(x, t; x, t - \tau) \end{array} \right),$$  \hspace{1cm} (7)

where $G^+ = -\gamma - F^+ \partial_x + [Q(1 - \lambda^2) + D(F^+ + \lambda \sqrt{Q/D})^2] \partial_x^2$ and $G^- = -\gamma - F^- \partial_x + [Q(1 - \lambda^2) + D(F^- + \lambda \sqrt{Q/D})^2] \partial_x^2$.

The quantities $P_+(x, t)$ and $P_-(x, t)$ are the probabilities at any time $t$ to find the barrier at the + or − configuration, respectively. The force $F = -\partial U(x, t) / \partial x$ fluctuates
between \( F^+ = -E_1/x|_{x=1/2} = -2E_1 \) and \( F^- = -E_2/x|_{x=1/2} = -2E_2 \). The initial condition is \( \sum_{i=1}^{2} P_i(x,0) = \delta(x) \) as we start with the particle at the bottom \( x = 0 \). The boundary conditions for the reflecting \((x = 0)\) and absorbing \((x = 1/2)\) boundary, respectively, are \( \partial_x P(x,t)|_{x=0} = 0 \) and \( \partial_x P(x,t)|_{x=1/2} = 0 \).

III. THE MEAN FIRST PASSAGE TIME WITH TIME DELAY

The MFPT is defined as \([25, 26]\)

\[
T_1(x) = -\int_0^\infty t\partial_t P_1^+(x,t;x\tau,t - \tau) dt = \int_0^\infty P_1^+(x,t;x\tau,t - \tau) dt,
\]

\[
T_2(x) = -\int_0^\infty t\partial_t P_1^{-}(x,t;x\tau,t - \tau) dt = \int_0^\infty P_1^{-}(x,t;x\tau,t - \tau) dt,
\]

where we only calculate the MFPT from \( x = 0 \) to \( x = 1/2 \), because the MFPT from \( x = 0 \) to \( x = 1/2 \) is far larger than that from \( x = 1/2 \) to \( x = 1 \).

Simultaneously considering the time delay effect on the MFPT, the equations of the MFPT for the FPE satisfy the following equations \([23–26]\]

\[
-\gamma - 2(1 + \tau)E_1\partial_x + [Q(1 - \lambda^2) + D(-2(1 + \tau)E_1 + \lambda\sqrt{Q/D})^2]\partial_x^2 T_1 + \gamma T_2 + 1 = 0,
\]

\[
-\gamma - 2(1 + \tau)E_2\partial_x + [Q(1 - \lambda^2) + D(-2(1 + \tau)E_2 + \lambda\sqrt{Q/D})^2]\partial_x^2 T_2 + \gamma T_1 + 1 = 0. (9)
\]

Generally speaking, one cannot get the exact expression of the MFPT. Usually the problem is studied via Monte Carlo simulation, direct numerical solution of the stochastic differential equation is carried out using the well-known methods \([29]\). However, we restrict our attention to the time delay effects on the MFPT of the simple model in the case where the midpoint of the barrier fluctuates between \( \pm E \) (that is \( E_1 + E_2 = 0 \) and \( E_1 = -E_2 = E \)). The MFPT for a particle over a fluctuating potential barrier that starts at the bottom \((x = 0)\) is \( T = \sum_{i=1}^{2} T_i(0) \); according to the above analysis, we can obtain

\[
T = \frac{2A_1}{4E^2(1 + \tau)^2 + \gamma(A_1 + A_2)} - \frac{1}{\gamma} + \left( \frac{2E + 2E\tau}{\gamma} h_1 - \frac{A_1 h_1^2}{\gamma} + 2 \right) a_1 + \left( \frac{2E + 2E\tau}{\gamma} h_1 - \frac{A_1 h_1^2}{\gamma} + 2 \right) a_2 + \frac{2E + 2E\tau}{\gamma} a_3 + 2a_4, (10)
\]

where

\[
A_1 = \frac{Q(1 - \lambda^2) + D(\lambda\sqrt{Q/D} - 2E - 2E\tau)^2}{\gamma},
\]

\[
A_2 = \frac{Q(1 - \lambda^2) + D(\lambda\sqrt{Q/D} + 2E + 2E\tau)^2}{\gamma}. (11)
\]

\[
h_1 = \frac{[2EA_2(1 + \tau) - 2EA_1]{\sqrt{(EA_1 + EA_2 + E\tau A_1 + E\tau A_2)^2 + \gamma A_1 A_2(A_1 + A_2)}/(A_1 A_2),}
\]

\[
h_2 = \frac{[2EA_2(1 + \tau) - 2EA_1]{\sqrt{(EA_1 + EA_2 + E\tau A_1 + E\tau A_2)^2 + \gamma A_1 A_2(A_1 + A_2)}/(A_1 A_2). (12)}
\[ a_1 = \frac{(b_2 c_3 - b_3 c_2)}{(b_1 c_2 - b_2 c_1)}, \]

\[ a_2 = \frac{-a_1 b_1 - b_3}{b_2}, \]

\[ a_3 = -a_1 h_1 - a_2 h_2, \]

\[ a_4 = -a_1 \exp h_1 - a_2 \exp h_2 - a_3 - \gamma /[-4E^2 - \gamma (A_1 + A_2)]. \]  

\[ b_1 = [2E(1 + \tau)h_1/\gamma - A_1 h_1^2/\gamma]h_1, \]

\[ b_2 = [2E(1 + \tau)h_2/\gamma - A_1 h_2^2/\gamma]h_2, \]

\[ b_3 = 4E(1 + \tau)/[-4E^2(1 + \tau)^2 - \gamma (A_1 + A_2)]. \]  

\[ c_1 = -2E(1 + \tau)h_1/\gamma + [2E(1 + \tau)h_1/\gamma - A_1 h_1^2/\gamma] \exp h_1, \]

\[ c_2 = -2E(1 + \tau)h_2/\gamma + [2E(1 + \tau)h_2/\gamma - A_1 h_2^2/\gamma] \exp h_2, \]

\[ c_3 = 4E(1 + \tau)/[-4E^2(1 + \tau)^2 - \gamma (A_1 + A_2)] - 2A_1 /[-4E^2(1 + \tau)^2 - \gamma (A_1 + A_2)] - 1/\gamma. \]  

Numerical integration is employed to calculate the MFPT; according to Eq. (7) we can calculate the flipping rate \( \gamma \) as a function of the MFPT. To present the relationship between the two variables better, we take their logarithms.

RA is defined as that MFPT of a particle driven by additive noises over a fluctuating potential barrier, which exhibits a minimum as a function of the potential barrier flipping rate. In Fig. 2, it is clear that the curves display minimums no matter whether accompanied with time delay or not. However, the RA phenomenon with time delay is obviously stronger than in the case of no time delay feedbacks. The time delay effect enhances RA for a positive correlation. With increasing noise correlation strength, the RA becomes more and more distinct in Fig. 3 for non-delay, i.e., the correlation between additive and multiplicative noise can enhance RA. This conclusion is the same as that in Ref. [14]. However, we try to emphasize the time delay feedbacks on the RA through comparing Fig. 3 with Fig. 4, which considers the time delay effects. When we take \( \tau = 0.2 \) in Fig. 4 evidently there is an interesting phenomenon, that is the so-called critical phenomenon. For the curves there exists a critical value \( \lambda = 0.02 \) which distinguishes the curves. The curves with values less than the critical value show the same RA phenomenon, but for the values larger than it they present the resonant restraint (RR) phenomenon. It is due to the combination of the time delay and the correlated noises that can change the steady properties of the system. Moreover, there is a time-delay induced critical behavior.

IV. DISCUSSION AND CONCLUSION

We have studied the time delay effects on the RA phenomenon over a fluctuating barrier driven by correlated noises. The results show that (1) time delay effects can enhance RA with the interplay of the correlation of the noises; (2) in the non-delayed case, with increasing the noise correlation intensity, the RA is strengthened; (3) in the synergistic
reaction of time delay and noise correlation there is a critical phenomenon which distinguishes resonant activation (RA) and resonant restraint (RR) by the correspond critical value $\lambda = 0.02$. Under the synergistic action of time delay and noise correlation, the steady properties of the system changed abundantly so that it showed some interesting phenomena, including critical behavior, resonant activation, and resonant restraint.
FIG. 4: The logarithm of the MFPT $\ln T$ as a function of the flipping rate $\gamma$ with $Q = 0.02$, $D = 0.02$, $E = 2$, and $\tau = 0.2$ fixed, where $\lambda$ takes the values $-0.4$, $-0.2$, $0.02$, $0.2$, $0.4$.

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References