Determining the Trajectory from Discrete RL Information

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We illustrate how to determine the control parameter $\lambda$ and the underlying $x_n$ for a logistic map from the RL-sequence. For a nonperiodic sequence, we can in principle determine $\lambda$ and $x_n$ to an arbitrarily high precision if we are given an infinite RL-sequence. For a finite sequence of size $N$, the limit of accuracy varies as $1/N^2$. The algorithm of finding $\lambda$ and $x_n$ is surprisingly simple.

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I. INTRODUCTION

For a simple-hump map $[1,2]$, 

$$x_{n+1} = f(x_n), \quad (1.1)$$

the mapping is 2 to 1. It is not possible to obtain the inverse image $x_n$ from $x_{n+1}$ unless we know whether $x_n$ is on the right (R) or left (L) of the peak. We shall refer this discrete set of right/left information as RL-sequence. For a periodic cycle, the RL-sequence gives rise to the U-sequence of the cycle. In a classic paper by Metropolis, Stein and Stein $[3,4]$, they showed that the order of appearance of a cycle is given completely by its U-sequence. In this paper, we wish to explore the information contained in the RL-sequence of a nonperiodic trajectory. In particular, we wish to determine the control parameter and the underlying $x_n$ of the map. Note that the RL-sequence is invariant under a conjugate transformation.

Two conjugate maps

$$f = h^{-1} \circ g \circ h \quad (1.2)$$

are related to each other by an invertible transformation.
\[ y_n = h(x_n), \quad x_n = h^{-1}(y_n). \]  

Under this transformation, the maps

\[ x_{n+1} = f(x_n) \quad \text{and} \quad y_{n+1} = g(y_n) \]  

transform into each other with peak into peak, R into Rand L into L. Hence, the equivalent maps will have the same RL-sequence. Thus, from a RL-sequence, we can only hope to determine a map up to a conjugate transformation. Since every single hump map is conjugate to a quadratic map with an appropriate parameter, we can restrict our construction to a family of quadratic maps. One familiar parametrization of quadratic maps is the logistic map,

\[ x_{n+1} = \lambda x_n(1 - x_n), \quad 0 < x_n < 1. \]  

The control parameter \( \lambda \) runs from 0 to 4. In the following, we shall restrict ourselves to the determination of \( \lambda \) and \( x_n \) as defined in (1.5).

II. TENT MAP AND THE TRUNCATED TENT MAP

As an intermediate step, it is more convenient to introduce a family of truncated tent maps. A tent map is defined by (Fig. la)

\[ x_{n+1} = 1 - 2|x_n|, \quad -1 < x_n < 1, \]  

and a truncated tent map is (see Fig. lb)

\[ x_{n+1} = \begin{cases} 
1 - 2|x_n| & \text{for } |x| > x_0 \\
1 - 2x_0 & \text{for } |x| < x_0.
\end{cases} \]  

The tent map is a special case of the truncated tent map with \( x_0 = 0 \). As one varies \( x_0 \), the truncated tent maps form a family of single-hump maps. Parameter \( x_{\text{max}} = 1 - 2x_0 \) plays the role of the control parameter. Strictly speaking, a truncated tent map is not really single-humped, but has a degenerated peak. However, we may treat the family of truncated tent maps as if they were single-humped for our construction purpose. We can associate each single-hump map with a member of the truncated tent maps.

We can use the tent map to determine whether a given RL-sequence is the U-sequence of a superstable cycle, and if it is, to decide the order of appearance of this cycle. The order of appearance of a cycle has been given in Metropolis, Stein and Stein [3]. The method presented here is equivalent to their method, but is more convenient to use.

From any RL-sequence, we construct the inverse images of \( x \equiv 0 \) in the tent map by

\[ x_n = \text{sign}(x_n)(1 - x_{n+1})/2, \]  

(2.3)
where the signs are provided by the RL-sequence. Let $N - 1$ be the length of the sequence. We start with $x_N = 0$ and compute $x_{N-1}, x_{N-2}, \ldots, x_1$.

**Lemma:** An RL-sequence is a U-sequence of an $N$ cycle if and only if $x_1$ is the largest among $x_n, n = 1, 2, \ldots, N$. The magnitude of $x_1$ determines the order of appearance of this $N$ cycle.

For the truncated tent map with $x_{\text{max}} = x_1$ given above, its attractor is an $N$ cycle with the given U-sequence.

### III. Determination of $\lambda$ and $X_N$

We shall determine $\lambda$ and $x_n$ in a logistic map from a set of RL-sequence. If the RL-sequence is repeating with length $L$, it describes either an $L$ cycle or a $2L$ cycle. From these repeating segments, one can construct only one U-sequence. The observed RL-sequence corresponds to the stable cycle associated with the U-sequence, or its bifurcation. We can determine the parameter $\lambda$ and the coordinates $x_n$ to within a range specified by the stability of the cycle. Since the sequence is repeating, we can extract only a limited amount of information from this RL-sequence no matter how long the sequence is. Thus, we can only determine $\lambda$ and $x_n$ to a limited accuracy.

The situation is more interesting if RL-sequence is nonperiodic. Here, we can in principle determine $\lambda$ and $x_n$ to an arbitrary accuracy if we have an arbitrarily long RL-sequence data. Consider, for instance, the sequence

$\text{RRRLRRRRRLRLRR}$
This sequence can be generated at $\lambda = 4$ but not by any value of $x$ at $\lambda < 2$. For $\lambda < 2$, only sequence $*L*L*$ is possible where $*$ may be either $R$ or $L$. From continuity argument, there must be a minimum $\lambda$ such that this sequence is possible. In the present case, this sequence can be generated by a minimum $\lambda = 3.6872$ and an initial $x = 0.838225$. It is known that if a RL-sequence exists in the logistic map with parameter $\lambda_0$, it also exists in all maps with $\lambda > \lambda_0$.

In the following, we shall examine a mathematical analog. Consider points generated randomly with equal probability inside a circle of radius $R$. To determine $R$ after $N$ points being generated, we construct a circle of minimal radius $r$ to contain all points. Obviously, $r$ is a lower bound of $R$. In principle, we can be unlucky and may encounter a situation where $r$ is significantly smaller than $R$ even for a large $N$. However, this is statistically very improbable. Let $\delta \equiv R - r$ be the error. We expect that $\delta$ goes to zero as $1/N$. The probability of having a small but finite error $\delta$ as $N \rightarrow \infty$ goes as $\exp(-\text{const}.N\delta)$. Thus, $r$ usually gives a very good approximation of $R$ at large $N$.

In this paper, a finite RL-sequence is given. This is the analog of a set of points inside a circle. We construct a minimal $\lambda$, called $\lambda_{\text{min}}$, and the associated $x_n$ which reproduce the given RL-sequence. Our $\lambda$ is the analog of $r$, and $\lambda_{\text{min}}$ is the analog of $r$. At large $N$, the $\lambda_{\text{min}}$ and the associated $x_n$ give good approximations of the real $\lambda$ and $x_n$. In our case as we shall see, the error goes as $1/N^2$. To facilitate the construction, we go through the intermediate step of constructing a minimal truncated tent map. The detailed steps of construction go as follow:

1. Construction of minimal truncated tent map

From any assigned value for the last $x_{N-1}$, we can iterate the tent map backward to obtain all values of $x_n$. Denote $x_{\text{max}}$ as the largest among $x_n$. Vary the last $x_N$ to achieve a minimum $x_{\text{max}}$. We now construct a truncated tent map with $x_0 = (1 - x_{\text{max}})/2$. It is obvious that if we start with this minimal $x_{\text{max}}$ and iterate the truncated tent map forward, we shall reproduce the RL-sequence to the right of this point. On the other hand, if we iterate the truncated tent map backward from $x_{\text{max}}$ according to the given RL-sequence, we shall never reach a larger $x_{\text{max}}$ nor $x_n$ in the region $(-x_0, x_0)$. Hence, we have succeeded to construct an entire range of $x_n$ of the truncated tent map obeying the given RL-sequence. It is also clear that no truncated tent maps with a smaller $x_{\text{max}}$ can produce the required RL-sequence. Our truncated tent map is the minimal tent map admitting the given RL-sequence.

2. Determination of $\lambda_{\text{min}}$ of the logistic map

As we compute $x_n$ from the inverse truncated tent map for a given RL-sequence, $x_n$ is a linear function of $x_N$. Hence, $x_n$ can not be a local minimum as a function of $x_N$. This implies that the minimum value of $x_{\text{max}}$ occurs only at the intersection of two
different $z_n$'s. The U-sequence of the truncated tent map for this minimal $x_{\text{max}}$ is the RL-sequence between these two $x_{\text{max}}$'s. Of course, we can also generate this U-sequence from the truncated map itself once we know $x_{\text{max}}$. We then obtain the parameter $\lambda$ in the logistic map to produce the same U-sequence. This gives rise the required minimal $\lambda$.

(3) Finding $x_n$ in the logistic map

At the minimal $\lambda$, we can determine the value of $x_n$ uniquely as follow: We take $x_{\text{max}}$ as the image of the peak (i.e. 0.5). We obtain other $x_n$ by either mapping forward or backward. In the backward map, we use the RL-sequence to resolve the sign ambiguity. The backward map will not lead to any imaginary $x_n$.

When we have a large RL-sequence (e.g. $N = 6400$), we can start with some small sequence at the end. For instance, we can find the minimal $\lambda$ and $x_n$'s based on the last 64 of the RL-sequence. Starting with the last $x_N$ determined by the last 64 of the sequence, we can iterate the tent map backward over the entire RL-sequence and obtain the true $x_{\text{max}}$ for the entire sequence. From the location of the true $x_{\text{max}}$, we read off the maximal $x_{\text{max}}$. This leads to a very efficient determination of $\lambda_{\text{min}}$ for the entire sequence. Knowing $\lambda_{\text{min}}$, we can start from $x_N$ of the logistic map based on the last 64 RL-sequence and iterate the logistic map backward over the entire sequence. This determines all $x_n$ of the logistic map.

There are intrinsic limitation for the determination of the values of $x_n$ near the end of the sequence. In the forward map, a small change can grow exponentially with iterations. As long as none of the R switches to L or L to R, we can not determine these changes. The little difference between the actual $\lambda$ and $\lambda_{\text{min}}$ is enough to introduce these uncertainties. This kind of uncertainties occurs only for $x_n$'s near the end of the RL-sequence.

IV. SOME ACTUAL CONSTRUCTION

We shall use the sequence RRRLRRRRRLRLRRRR as an example. Here, $N = 16$. We first consider the tent map. For any chosen $x_{16}$, we can use the RL-sequence and the inverse tent map to obtain all $x_n$. Let $x_{\text{max}}$ be the maximum of these $x_n$. Vary $x_{16}$ to obtain a minimum $x_{\text{max}} = 0.668626$. This occurs at positions $x_3$ and $x_{12}$. We then break the RL-sequence into an upper sequence RR and a lower sequence RLRRRRRLRLRRRR. We can identify the U-sequence of the truncated tent map from the lower sequence as RLRRRRRR. This identification enables us to obtain the minimal parameter for the logistic map as

$$\lambda_{\text{min}} = 3.687216.$$  

Taking $\lambda$ as $\lambda_{\text{min}}$, we can iterate the logistic map both forward and backward from $x_2 = 0.5$ to get all $x_n$.  

The RL-sequence given above is actually the last 16 letters of a much larger data base. We obtain this data base by choosing an arbitrary \( \lambda \), a random initial \( x \), and use the logistic map to generate several thousands of \( R \) and \( L \). We throw away the first 500 of these letters and keep the next 6400 of them. If we increase the size of the sequence used in our computation, we get a better determination of \( \lambda \). In the following, we take the last \( N \) letters from the data base, and obtain the results below:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>3.687216</td>
</tr>
<tr>
<td>32</td>
<td>3.699317</td>
</tr>
<tr>
<td>64</td>
<td>3.699317</td>
</tr>
<tr>
<td>128</td>
<td>3.699608</td>
</tr>
<tr>
<td>256</td>
<td>3.699997</td>
</tr>
<tr>
<td>6400</td>
<td>3.69999959</td>
</tr>
</tbody>
</table>

The exact value of \( \lambda \) is 3.7. As we can see, the \( \lambda \)'s obtained above are indeed the lower bounds of the exact value. The difference of the determined \( \lambda \) and the true \( \lambda \) goes to zero roughly as \( 1/N^2 \). Note that as \( N \) goes from 32 to 64, the calculated \( \lambda \) does not change at all. This is because that the highest U-sequence imbeded in the last 64 letters has already appeared in the last 32 letters. For \( N = 6400 \), the error is only \( 4 \times 10^{-7} \). Knowing \( \lambda \), we can determine the underlying \( x \). Except for the \( z_n \)'s near the end of the sequence, the difference between the reconstructed \( z_n \) and the original \( z_n \) is of the order of \( 10^{-7} \). The agreement of the calculated \( \lambda \) and \( z_n \) with the original \( \lambda \) and \( z_n \) are excellent for all nonperiodic sequences. Nonperiodic sequence begins at the bifurcation limiting points (Feigenbaum point) at \( \lambda = 3.569947 \). Even at this \( \lambda \), the error remains to be small.

V. DISCUSSION

We illustrate here that for a single-hump iterative map, one can determine both the control parameter and the underlying values of a map from a RL-sequence. The dynamics of RL-sequence is known as symbolic dynamics. A survey of symbolic dynamics for a two dimensional Henon-type map may be found in P. Cvitanovic et al. [5]. It is interesting to find out whether one can determine the control parameter and the underlying values of a two-dimensional map from some discrete symbolic information. Other possibility is to study the trajectory of a hamiltonian system from some discrete information. One such system is a particle moving inside a triangle. Let \( a \), \( b \), \( c \) denote the sides of the triangle. We can associate each of the particle trajectories with a sequence of letters describing the order of sides that the trajectory meets. It is still an open question whether one can determine a nonperiodic trajectory completely from its sequence of letters alone.
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REFERENCES

[4] References 1, 3 and many other excellent articles are included in the reprint books by P. Cvitanovic, Universality in Chaos (Adam Hilger 1984), and Hao Bai-Lin, Chaos II (World Scientific 1990).