Note on Second Born Approximation and Proton-Neutron and Proton-Proton Scattering

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The scattered amplitude $f(d)$ is regarded as a series in powers of the interaction potential $V$, and the second Born approximation consists in calculating the scattered intensity up to the fourth power of $V$. The calculation can be carried out analytically for a Gaussian potential, leading to a simple way of calculating the second approximation for the phases. The result is applied to the proton-neutron and proton-proton scattering at 100 MeV, on the basis of the three forms of nucleon interaction suggested by Rata and Schwinger, with a Gaussian $V(r)$ and without tensor forces. The result for the symmetrical theory is:

$I_+ = 0.96 \times 10^{-15}$ cm$^2$, $I_- = 0.22 \times 10^{-15}$ cm$^2$, compared with the first approximation values $1.40 \times 10^{-15}$ and $0.35 \times 10^{-15}$ cm$^2$, respectively. These give for $(I_+ + I_-)$ the value 1.18, in much closer agreement with the observed value 1.17 for $I_{tot}$ than the other two forms of interaction potential.

I. INTRODUCTION

For the scattering by a central field, the scattered amplitude is given by the Faassen-Holtzmark formula, namely,

$$f(d) = \frac{1}{2k} \sum (2l+1)(2l+1) P_l(\cos d)$$

The usual first Born approximation consists in (i) dropping off all terms except $\delta_1$, and (ii) replacing $\delta_1$ by its first approximation

$$\delta_1^{(0)} = \frac{2\lambda}{2k^4} \int \psi^* \cdot \psi \frac{d^4r}{|r|}$$

where $\lambda$ is the proton or neutron mass for proton-neutron collisions.

Recent calculations on the cross section of proton-neutron scattering at high energies show that the first Born approximation is unsatisfactory at 100 MeV. As the exact calculation of the phases $\delta_1$ by numerical integration of the wave equation is lengthy for any potential other than the rectangular hole one, it is of some interest to have better approximate methods than the usual Born approximation.

One usual method of improving on the Born approximation is to use $\delta_1^{(0)}$ in (I), so as to take into account the higher powers of $\delta_1$. This procedure will be justified if the difference $\delta^{(0)} - \delta_1^{(0)}$ is very small compared with $(\delta_1^{(0)})^2$. This is, however, in general not the case for low values of $I$, so that this procedure does not form any consistent approximation in the sense of the perturbation theory. It has been shown that the phase $\delta_1$ can be developed as a series in powers of the interaction potential $V(r)$, namely,

$$\delta_1 = \delta_1^{(0)} + \delta_1^{(1)} + \cdots$$

where $\delta_1^{(0)}$ is given by (2) and

$$\delta_1^{(1)} = (-1)^r \left( \frac{2\lambda}{2k^4} \right) \int \psi^* \cdot \psi r^{2k^2} dr$$

Calculation of $\delta_1^{(1)}$ by means of this expression is very lengthy, and is not difficult.

1 Tzu-Yow Wu, Phys. Rev. 73, 973 (1948); M. Carmack and H. A. Barbe, Phys. Rev. 73, 191 (1948).
Another approximate formula for $\delta_1$ has been found by Pais\(^3\) by the variational method. The expression is

$$\frac{2\ell + 1 - (2\ell + \ell')}{2\ell + 1 - (4\ell + \ell')} = \frac{1}{4M} \int_{-\infty}^{\infty} V(r) r^2 \left( e^{i\ell r} - e^{i\ell' r} \right) dr. \quad (5)$$

If $\delta_1$ is small, this can be put in the form (3), with $\delta_1^{(2)}$ given by (2) and

$$\delta_1^{(2)} = \frac{2}{\sqrt{m}} \left( a_i - \frac{\ell (\ell + 1)}{2\ell + 1} \right) \delta_1^{(1)}, \quad (6)$$

where

$$\delta_1^{(1)} = - \left( \frac{\partial \delta_1^{(1)}}{\partial \rho} \right)_{\rho=0}. \quad (6a)$$

When $\delta_1$ is small, calculation of $\delta_1^{(2)}$ according to (6) is easy. Unfortunately, Pais' method is not valid for low values of $\ell$, in particular, and recourse must be made to numerical solution of the wave equation.

In the present note, we shall obtain the second Born approximation and apply the result to proton-neutron and proton-proton scatterings at 100 MeV.

### IL SECOND BORN APPROXIMATION

We shall regard $f(\theta)$ as a series in powers of the interaction potential $V(r)$. The scattered intensity $I(\theta)$ is, up to the fourth power of $V(r)$,

$$I(\theta) = \left| f^{(1)}(\theta) \right|^2 + \left| f^{(2)}(\theta) \right|^2 + 2 f^{(1)}(\theta) f^{(2)}(\theta), \quad (7)$$

where

$$f^{(1)}(\theta) = \frac{1}{k} \sum (2\ell + 1) \delta_1 P_\ell (\cos \theta), \quad (8a)$$

$$f^{(2)}(\theta) = \frac{1}{k} \sum (2\ell + 1) \delta_1 P_\ell (\cos \theta), \quad (8b)$$

$$f^{(3)}(\theta) = - \frac{2}{3k} \sum (2\ell + 1) \delta_1 P_\ell (\cos \theta). \quad (8c)$$

In the second and the third term in (7), it is sufficient to employ $\delta_1^{(1)}$ in (8), but in the first term it is necessary to employ $\delta_1^{(1)} + \delta_1^{(2)}$.

$$\delta_1 = \delta_1^{(1)} + \delta_1^{(2)}. \quad (9)$$

Instead of summing (8a) with $\delta_1 = \delta_1^{(1)} + \delta_1^{(2)}$ and $\delta_1^{(2)}$ given by (4) or a similar expression obtained by the perturbation method, it is found convenient to calculate the first two terms in (7) together, namely,

$$\left| f^{(1)}(\theta) \right|^2 + \left| f^{(2)}(\theta) \right|^2 = \left| f^{(1)}(\theta) + f^{(2)}(\theta) \right|^2.$$

Let $\psi$ be the solution of

$$\Delta \psi + \frac{M}{2\hbar^2} V(r) \psi = 0,$$

and let

$$\psi = \psi_{i1} + \psi_{i2} + \psi_{i3} + \psi_{i4},$$

where $\psi_{i4}$ is the solution of the equation

$$\Delta \psi + \frac{M}{\hbar^2} \psi = 0,$$

and represents the incident wave. By successive approximation, one obtains

$$\left[ \Delta + \frac{M}{\hbar^2} V(r) \right] \psi_{i1} = \frac{M}{2\hbar^2} V(r) \psi_{i4},$$

and their solutions

$$\psi_{i1}(r) = \frac{1}{4\pi} \int \frac{e^{i(kr - \mathbf{r}'r')}}{r' - r} \frac{M}{\hbar^2} V(r') \psi_{i4}(r') \, dr', \quad \psi_{i3}(r) = \frac{1}{4\pi} \int \frac{e^{i(kr - \mathbf{r}'r')}}{r' - r} \frac{M}{\hbar^2} X V(r') \psi_{i4}(r') \, dr'.$$

From the asymptotic solutions $\psi_{i1}(r), \psi_{i2}(r)$ for large $r$, one obtains the scattered amplitude up

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to the second power of I,
\[ f(\theta) = f_0(\theta) + f_\tau(\theta) \]
\[ = -\frac{M}{4\pi k^3} \int e^{-i|\mathbf{r}' - \mathbf{r}|} V(\mathbf{r}') e^{i\mathbf{k} \cdot \mathbf{r}' - i\mathbf{k} \cdot \mathbf{r}} d^3 r' \]
\[ + \left( \frac{\hbar}{4\pi k^3} \right)^{-1} \int e^{-i|\mathbf{r}' - \mathbf{r}|} \frac{V(\mathbf{r}') e^{i\mathbf{k} \cdot \mathbf{r}' - i\mathbf{k} \cdot \mathbf{r}}}{|\mathbf{r}' - \mathbf{r}|} \times V(\mathbf{r}) e^{i\mathbf{k} \cdot \mathbf{r}} d^3 r'. \quad (10) \]

Now it is seen that
\[ f_0(\theta) + f_\tau(\theta) = f^0(\theta) + f^{\tau}(\theta). \quad (11) \]

Since \( f_\tau(\theta) \), the first term in (10), is known to be the sum of the first and imaginary part of \( f(\theta) \) in (10) must be, respectively, real part of
\[ f_\ell(\theta) = \sum_k (2\ell + 1) \tilde{C}^{(\ell)} P_\ell(\cos \theta), \quad (12a) \]
imag. part of
\[ f_{\ell}(\theta) = \sum_k r(2\ell + 1)(\tilde{C}^{(\ell)} P_\ell(\cos \theta)), \quad (12b) \]

To calculate the last term \( f^{\ell}(\theta) f^{\ell}(\theta) \) in (7), one would have to sum (8a) with \( \delta_k \) replaced by \( \delta_0 \). This series cannot be summed in a simple manner as (12b). However, as the \( \delta_k, \delta_0, \delta_2, \ldots \) must converge, we may approximate \( f(\theta) \) by taking the first few terms, say three, in (12a). Up to the approximation desired, the \( f_0(\theta) \) in the last term in (7) is given by the usual Born expression.

To enable \( f_\ell(\theta) \) to be evaluated in analytical form and to demonstrate the relation (12b), we shall take a Gaussian potential \( V(\mathbf{r}) = V_r \exp(-a^2 \mathbf{r}^2) \).

The relation (12b) in this case can then be proved by means of Weber's second exponential integral together with the addition theorem of Bessel functions. For Gaussian potential,
\[ f_\ell(\theta) = \frac{\sqrt{V_r}}{2} \frac{1}{a^2} \frac{\sin \theta}{\sin \theta_{\ell}} \frac{1}{r(2\ell + 1) \cos \theta} \]
for ordinary and Majorana exchange force, respectively. For ordinary force, one readily finds
\[ \begin{align*}
2\pi g \int_{0}^{\pi} f^{\ell}(\theta) f^{\ell}(\theta) \sin \theta d\theta &= -\frac{15\pi a}{2a^2} \left( \frac{M V_r}{4a^2 k^3} \right) \left[ \frac{5}{2} \left( \frac{1 - \exp(\frac{k^2}{2})}{\frac{k^2}{2}} \right) \right] \\
&\quad + \frac{3(3\ell + 1) - 15}{2a^2} \left[ 1 - \frac{a^2}{k^2} - \frac{1}{\frac{k^2}{2}} \exp \left( \frac{k^2}{2} \right) \right] \exp \left( \frac{k^2}{a^2} \right) + \cdots \quad (16)
\end{align*} \]
For the case of exchange force, one employs the lower expression in (15) and remembers that the $\delta_{1}$ with odd $I$ change their sign. Hence

$$\partial f_{1}(\theta), f_{2}(\theta) |_{\text{exchange}} = \partial f_{1}(\theta = 0), f_{2}(\theta = 0) |_{\text{ordinary}},$$

and the contribution to the total cross section from the last term in (7) is again given by (16).

For interaction potentials which are mixtures of the ordinary and exchange force, $f_{\text{m}}(\theta), f_{\text{m}}(\theta)$ can be found in a similar manner, as will be illustrated below in II.

A better approximation than the second can be obtained by substituting (9), namely, $\delta_{1} = \delta_{1}^{(0)} + \delta_{1}^{(m)}$ into (13) so as to include the contribution of all powers of $\delta_{1}$ higher than the fourth. While this procedure again does not form any definite approximation in the sense of the perturbation theory, it may be justified and expected to be very good if

$$\delta_{1}^{(m)} = (\delta_{1}^{(0)} + \delta_{1}^{(m)}) \ll (\delta_{1}^{(0)}).$$

To obtain the $\delta_{1}^{(0)}$, one may either calculate them by the perturbation theory, by (4), by (6), or, in the case of the Gaussian potential, make use of the following procedure which is very much shorter. From (12a) and (14), one has

$$\sum (2 + 1) \partial f_{1}(\theta), f_{2}(\theta) |_{\text{cos}} = \frac{1}{4a^{2}h} \exp \left(- \frac{\theta^{2}}{2a^{2}} \right) \exp \left(- \frac{\theta^{2}}{2a^{2}} \right) \frac{\partial}{\partial \theta} \left\{ e^{x} \theta^{x} \right\}^{1/2} \left[ e^{-x} \theta^{x} \right] \left[ e^{-x} \theta^{x} \right] \left[ e^{-x} \theta^{x} \right] \left[ e^{-x} \theta^{x} \right].$$

By taking different values of $\theta$, one can calculate the first $\delta_{1}^{(0)}$ which are not negligible.

### III. PROTON-NEUTRON AND PROTON-PROTON SCATTERING

We shall apply the above result to the problem of proton-neutron and proton-proton scattering at 100 Mev on the basis of the three forms of nucleon interaction suggested by Rarita and Schwinger with Gaussian dependence on $r$, and without tensor force, namely,

- Neutral
  $$V(r) = \frac{1}{2} \left( 1 - \frac{r}{a} \right) \exp(-\frac{r^{2}}{a^{2}}).$$

- Charged
  $$V(r) = \frac{1}{2} \left( 1 + \frac{r}{a} \right) \exp(-\frac{r^{2}}{a^{2}}).$$

- Symmetrical
  $$V(r) = \frac{1}{2} \left( 1 - \frac{r}{a} \right) \exp(-\frac{r^{2}}{a^{2}}).$$

We have chosen the following constants

$$V_{0} = 4.5 \text{ Mev}, \quad \frac{1}{2} c_{\text{m}} = 26 \text{ Mev}, \quad \frac{1}{2} c_{\text{m}}^{2} = 0.266 \times 10^{-17} \text{ cm}^{-2}$$

to fit the data on the ground state of the deuteron and the proton-proton scattering at low energies. The scattered amplitudes $f_{1}^{(1)}(\theta), f_{1}^{(2)}(\theta), f_{1}^{(3)}(\theta)$ in the case of the "N" and the "C" theory can be readily obtained as explained in II. For the symmetrical theory, it can be shown that for the triplet state scattering,

$$f_{1}^{(1)}(\theta) = \frac{1}{2} \left( 1 + \frac{r}{a} \right) \exp(-\frac{r^{2}}{a^{2}}),$$

$$f_{1}^{(2)}(\theta) = \frac{1}{2} \left( 1 - \frac{r}{a} \right) \exp(-\frac{r^{2}}{a^{2}}),$$

$$f_{1}^{(3)}(\theta) = \frac{1}{2} \left( 1 + \frac{r}{a} \right) \exp(-\frac{r^{2}}{a^{2}}).$$

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and for the singlet state scattering,

\[ V(r) = -(1 - 2g) + \frac{1}{2} Z \alpha V exp(-\alpha r^2), \]

\[ f_{\alpha}^{(s)}(d) = \left(1 - 2g\right) \left(1 - f_{\alpha}^{(s)}(d) + 2f_{\alpha}^{(s)}(d) \right), \]

\[ f_{\alpha}^{(s)}(d) = \left(1 - 2g\right) \left(3f_{\alpha}^{(s)}(d) - f_{\alpha}^{(s)}(d) \right), \]

where \( f_{\alpha}^{(s)}(d) \) is given by (14), and \( f_{\alpha}^{(s)}(d) \) are given by the upper and lower expression in (13), respectively. For \( f^{(s)}(d) \), the appropriate phases \( \delta_1 \) are given in Table I.

For the potential (13) with constants as given in (20), we obtain:

\[ V(r) = -(1 - 2g) + \frac{1}{2} Z \alpha V exp(-\alpha r^2), \]

\[ f_{\alpha}^{(s)}(d) = \left(1 - 2g\right) \left(1 - f_{\alpha}^{(s)}(d) + 2f_{\alpha}^{(s)}(d) \right), \]

\[ f_{\alpha}^{(s)}(d) = \left(1 - 2g\right) \left(3f_{\alpha}^{(s)}(d) - f_{\alpha}^{(s)}(d) \right), \]

where \( f_{\alpha}^{(s)}(d) \) is given by (14), and \( f_{\alpha}^{(s)}(d) \) are given by the upper and lower expression in (13), respectively. For \( f^{(s)}(d) \), the appropriate phases \( \delta_1 \) are given in Table I.

Table II. Differential cross section \( 2\pi f^{(s)}(d) \) in \( 10^{-8}\text{cm}^2\) of proton-neutron scattering in 100 MeV.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>1st Born</th>
<th>2nd Born</th>
<th>1st Born</th>
<th>2nd Born</th>
<th>1st Born</th>
<th>2nd Born</th>
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<td>1.083</td>
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<td>0.849</td>
<td>0.692</td>
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<td>0.0394</td>
<td>0.124</td>
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Table III. Total cross section in \( 10^{-8}\text{cm}^2\) of proton-neutron scattering at 100 MeV.

<table>
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<tr>
<th>( \theta )</th>
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</tbody>
</table>

Table IV. Twice the total cross section in \( 10^{-8}\text{cm}^2\) of proton-proton scattering at 100 MeV.

<table>
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<tr>
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<th>2nd Born</th>
<th>1st Born</th>
<th>2nd Born</th>
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</tr>
</tbody>
</table>

This is of interest to compare the phases (22) with those calculated by the method of Pais. They are as follows: \( \delta_2 = -0.34 \), \( \delta_2 = -0.34 \), \( \delta_2 = -0.34 \). For \( \delta_2 = -0.34 \), the method of Pais is not valid, but calculation with \( \delta_2 = -0.34 \), with the sign correct.
mined by the variational method, namely,

\[ \psi_{\text{var}} = -\frac{3}{2} (A + B) \frac{e^{-r}}{r} \]

\[ \psi_{\text{rel}} = (A + B) \frac{e^{-r}}{r} \]

\[ \psi_{\text{sym}} = (A - B) \frac{e^{-r}}{r} \]

\[ \psi_{\text{rect}} = -\frac{3}{2} (A - B) \frac{e^{-r}}{r} \]  

(23)

where

\[ A = -1.303, \quad B = 4.606, \quad V = 0.977 \times 10^{-8}, \quad X = 5.655 \times 10^{-7}. \]

We have also calculated the cross sections on the symmetrical theory in (19), replacing the Gaussian potential by a rectangular hole with the following constants

range \( r = 2.80 \times 10^{-10} \text{ cm} \), \( V_0 = 21 \text{ Mev} \), \( (1-2g) V_a = 11.7 \text{ Mev} \).

(24)

For this potential we have also calculated the exact \( \sigma \) for proton-neutron scattering. The result is given in Tables I and IV. A comparison of the values obtained for the Yukawa potential in (23) and the rectangular hole potential (24) with those obtained for the Gaussian potential leads one to think that a higher approximation for these potentials probably gives approximately the same values as for the Gaussian potential.

In view of the strong dependence of the total cross section on the proportion of ordinary and exchange force and comparatively weak dependence on the exact form of the radial \( V(r) \),

it seems of significance to compare the total cross section calculated with Gaussian potential for various mixtures of ordinary and exchange force with the experimental values. Recently Cook \( \psi_{\text{var}} \) reported the following total cross sections for 90 Mev:

\[ \sigma_{\text{var}} = 0.53 \times 10^{-25} \text{ cm}^2 \]

\[ \sigma_{\text{rel}} = 1.17 \times 10^{-25} \text{ cm}^2 \]  

(35)

It is seen that the calculated \( \sigma_{\text{var}} \) on the symmetrical theory agrees better with the observed value than the other two theories. On regarding the proton-deuteron cross section as approximately the sum of the proton-neutron and proton-proton cross sections, one finds for the ratio \( \sigma_{\text{var}} / \sigma_{\text{rel}} \) the values 0.36, 0.51, 0.69 for the neutral, charged, and the symmetrical theory, respectively, as compared with the observed value 1.17.

This agreement, however, does not establish the symmetrical theory in the form (19). The great difference between the observed value for the ratio \( \sigma_{\text{var}} / \sigma_{\text{rel}} \) for proton-neutron and the calculated value shown in Table I seems to indicate the presence of tensor force, whose effect is to increase the scattered intensity in directions \( \phi = \pi/2 \) (in the center of mass system). An exact calculation on the symmetrical theory, including tensor force, still leads to a much larger value for the ratio than the observed one. It seems that both the range of the force and the proportion of central and tensor forces have to be readjusted in order to agree with the meagre data now available at 90 Mev.

The writer wishes to express his indebtedness to Professor G. E. Uhlenbeck for helpful discussions.


Footnotes:


2 T. Y. Wu and J. Ashkin, Phys. Rev. 73, 986 (1948).