This paper presents the existence of Šil’nikov heteroclinic orbits in the Schimizu-Morioka system and the Zhou system by using the undetermined coefficient method. As a result, the Šil’nikov criterion along with some technical conditions guarantees that the Schimizu-Morioka and Zhou systems have both Smale horseshoes and horseshoe type of chaos. Moreover, the geometric structures of attractors are determined by these heteroclinic orbits.

I. INTRODUCTION

Homoclinic and heteroclinic orbits arise in the study of bifurcation and chaos phenomena as well as in their applications, as in mechanics, biomathematics, and chemistry [1, 2]. In some cases it is necessary to determine the nature or the type of chaotic behavior resulting from a dynamical system; one of the commonly agreed analytic criteria for proving chaos in autonomous systems is the work of Šil’nikov [3, 4], the resulting chaos is called horseshoe type or Šil’nikov chaos [5–8].

Since the discovery of the famous Lorenz chaotic system [9], researchers often think of seeking some kind of canonical forms for all possible continuous time quadratic autonomous chaotic systems in three dimensions. The existence of heteroclinic and homoclinic orbits of quadratic autonomous chaotic systems in three dimensions has been frequently discussed in academic investigations, such as for the Lorenz family system [10], the generalized Lorenz canonical form of dynamics system [11], the Chen system [12], the Lu system [13–15], the Liu system [16], the coupled Duffing’s systems [17], the modified Lorenz system [18], the new chaotic systems [19, 20], and the Genesio system [21] among several others based on the Šil’nikov criterion [3, 4].

The undetermined coefficient method is a powerful tool for determining the heteroclinic and homoclinic orbits of chaotic systems; it plays a major role in chaotic dynamics analysis as well as chaos control and synchronization. In this work, using the undetermined

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coefficient method a rigorous proof is introduced to prove the existence of Ši’lnikov orbits in two different chaotic systems, more exactly in the Schimizu-Morioka system and in the Zhou system. Both systems have exactly two heteroclinic orbits which are symmetrical with respect to the Z-axis. We prove that the Schimizu-Morioka system has only one type of Ši’lnikov orbit, i.e., heteroclinic orbits, but the Zhou system has two types of orbits, i.e., both heteroclinic and homoclinic orbits. Moreover, by applying the Ši’lnikov theorem, which provides an important theoretical criterion for proving the existence of a chaotic attractor, we are convinced that the two systems indeed are chaotic, with Smale horseshoes and horseshoe type of chaos.

This paper is organized as following: in Section II, some basic concepts and terminology related to homoclinic and heteroclinic orbits are reviewed. Section III is devoted to investigation of the Schimizu-Morioka system structure. In Section IV, the Ši’lnikov heteroclinic orbits of the Schimizu-Morioka system are studied in detail by using the undetermined coefficient method. In this section, the algebraic expression of the heteroclinic orbit will also be derived, and the uniform convergence of its series expansion is proved. Section V is devoted to investigation of the Zhou system structure. In Section VI the Ši’lnikov chaos of the Zhou system will be studied by using the undetermined coefficient method. In this section, the algebraic expression of homoclinic and heteroclinic orbits will be derived, and the uniform convergence of its series expansion is proved. Finally, some concluding remarks will be provided in Section VII.

II. HOMOCLINIC AND HETEROCLINIC ORBITS

Consider the third-order autonomous system
\[ \frac{dx}{dt} = f(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3 \]  
(1)

where the vector field \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) belongs to class \( C^r \) (\( r \geq 2 \)).

Let \( x \in \mathbb{R}^3 \) be the state variable of the system (1), and \( t \in \mathbb{R} \) be the time. Suppose that \( f \) has at least one equilibrium point \( E \). The point \( E \) is called a hyperbolic saddle focus for system (1) if the eigenvalues of the Jacobian \( A = Df(E) \) are \( \alpha, \rho \pm i\omega \) where \( \rho \alpha < 0, \ \omega \neq 0 \).

A homoclinic orbit \( \gamma(t) \) refers to a bounded trajectory of system (1) that is doubly asymptotic to an equilibrium point \( E \) of the system, i.e., \( \lim_{t \to +\infty} \gamma(t) = \lim_{t \to -\infty} \gamma(t) = E \).

A heteroclinic orbit \( \delta(t) \), is similarly defined, except that there are two distinct saddle-foci, \( E_1 \) and \( E_2 \), being connected by the orbit, one corresponding to the forward asymptotic time and the other to the reverse asymptotic time limit, \( \lim_{t \to +\infty} \delta(t) = E_1 \) and \( \lim_{t \to -\infty} \delta(t) = E_2 \).

The heteroclinic and the homoclinic Ši’lnikov criteria for the existence of chaos are summarized in the following two theorems [3, 4].

Theorem 1. The heteroclinic Ši’lnikov theorem. Suppose that two distinct equilibrium points, denoted by \( \chi^1 \) and \( \chi^2 \), respectively, of system (1) are saddle foci, whose characteristic
values $\gamma_k$ and $\rho_k \pm i\omega_k$ ($k = 1, 2$) satisfy the following Ši’lnikov inequalities:

$$|\gamma_k| > |\rho_k| > k = 1, 2, \quad \omega \neq 0$$

(2)

under the constraint

$$\rho_1\rho_2 > 0 \text{ or } \gamma_1\gamma_2 > 0,$$

(3)

suppose also that there exists a heteroclinic orbit joining $\chi_1^e$ and $\chi_2^e$, then:

(i) The Ši’lnikov map, defined in a neighborhood of the heteroclinic orbit, has a countable number of Smale horseshoes in its discrete dynamics;

(ii) For any sufficiently small $C^1$-perturbation $g$ of $f$, the perturbed system

$$\frac{dx}{dt} = g(x), \quad x \in \mathbb{R}^3$$

(4)

has at least a finite number of Smale horseshoes in the discrete dynamics of the Ši’lnikov map defined near the heteroclinic orbit;

(iii) Both the original system (1) and the perturbed system (2) have horseshoe type of chaos.

**Theorem 2.** the homoclinic Ši’lnikov theorem. Suppose that one equilibrium point of system (1), denoted by $\chi_e$, is a saddle focus, whose eigenvalues $\gamma$ and $\rho \pm i\omega$ satisfy the following Ši’lnikov condition:

$$\gamma\rho < 0, \quad |\gamma| > |\rho| > 0, \quad \omega \neq 0$$

(5)

suppose also that there exists a homoclinic orbit connecting $\chi_e$. Then:

(i) The Ši’lnikov map, defined in a neighborhood of the homoclinic orbit of the system, possesses a countable number of Smale horseshoes in its discrete dynamics;

(ii) For any sufficiently small $C^1$-perturbation $g$ of $f$, the perturbed system

$$\frac{dx}{dt} = g(x), \quad x \in \mathbb{R}^3$$

(6)

has at least a finite number of Smale horseshoes in the discrete dynamics of the Ši’lnikov map defined near the homoclinic orbit;

(iii) Both the original system (1) and the perturbed system (3) exhibit horseshoe type of chaos.

**III. STRUCTURE OF THE SCHIMIZU-MORIOKA SYSTEM**

The Schimizu-Morioka system [22, 23] can be described by the following system of differential equations:

$$\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= x - \lambda y - xz, \\
\frac{dz}{dt} &= -\alpha z + x^2
\end{align*}$$

(7)
where $\alpha, \lambda \in R^+$; this system exhibits a chaotic attractor at $\lambda = 0.75, \alpha = 0.45$.

The Schimizu-Morioka system (7) has three equilibrium points: $E_1 = (0, 0, 0)$, $E_{2,3} = (\pm \sqrt{\alpha}, 0, 1)$.

\[ \mu^3 + (\alpha + \lambda)\mu^2 + (z + \alpha \lambda - 1)\mu + 2x^2 - \alpha + \alpha z = 0. \] (8)

The characteristic equation at the equilibrium points $E_1$ is:

\[ \mu^3 + (\alpha + \lambda)\mu^2 + (\alpha \lambda - 1)\mu - \alpha = (\mu + \alpha)(\mu^2 + \lambda \mu - 1) = 0. \] (9)

The characteristic equation (9) has three real roots, $(-\alpha, \frac{-\lambda \pm \sqrt{\lambda^2 + 4}}{2})$ therefore $E_1$ is not a saddle focus. Then there is no homoclinic or heteroclinic orbits of Šil’nikov type.

The characteristic equation at the equilibrium points $E_2$ and $E_3$ is

\[ \mu^3 + (\alpha + \lambda)\mu^2 + \alpha \lambda \mu + 2\alpha = 0. \] (10)

Due to Descartes’ rule of signs [24, 25] the characteristic equation (10) has no positive real root. Thus it has at least one negative real root.

In Equation (10) Let $\mu = \eta - \frac{(\alpha + \lambda)}{4}$, then Equation (10) becomes:

\[ \eta^3 + p\eta + q = 0, \] (11)

where $p = \alpha \lambda - \frac{(\alpha + \lambda)}{3}$, $q = \frac{2(\alpha + \lambda)^3}{27} - \frac{\alpha \lambda(\alpha + \lambda)}{3} + 2\alpha$, and $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$.

By the Cardan formula, Equation (11) has a unique negative real root, $\alpha$, and a conjugate pair of complex roots, $\rho \pm i\omega$, with

\[ \gamma = \sqrt[3]{\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}}, \quad \rho = \frac{1}{2} \left(\sqrt[3]{\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}}\right), \]

and $\omega = \frac{\sqrt{3}}{2} \left(\sqrt[3]{\frac{q}{2} + \sqrt{\Delta}} - \sqrt[3]{\frac{-q}{2} - \sqrt{\Delta}}\right)$. 

FIG. 1: Phase portraits of the Schimizu-Morioka system in three dimensions.
When $\Delta > 0$, therefore, the algebraic Equation (10) has the following three roots:

$$
\mu_1 = -\frac{(\lambda + \alpha)}{3} + \gamma, \quad \mu_2 = -\frac{(\lambda + \alpha)}{3} + \rho + \omega i, \quad \mu_3 = -\frac{(\lambda + \alpha)}{3} + \rho - \omega i,
$$

(12)

respectively, where $\mu_1 < 0$, to ensure that the real part of the complex conjugate roots is positive, and it is further required that

$$
-\sqrt{-\frac{q}{2} + \sqrt{\Delta}} - \sqrt{-\frac{q}{2} - \sqrt{\Delta}} > \frac{2(\alpha + \lambda)}{3} > 0.
$$

(13)

Therefore, the equilibrium points $E_2$ and $E_3$ are saddle foci at the same time.

For $\alpha = 0.45$ and $\lambda = 0.75$ the equilibrium and their eigenvalues are given by:

$\begin{cases}
E_1 = (0, 0, 0), \quad (\lambda_1, \lambda_2, \lambda_3) = (-0.45, 0.693, -1.443), \\
E_2 = (0.6708, 0, 1), \quad (\lambda_1, \lambda_2, \lambda_3) = (-1.412264873, 0.1061324363 \pm 0.791208019i), \\
E_3 = (-0.6708, 0, 1), \quad (\lambda_1, \lambda_2, \lambda_3) = (-1.412264873, 0.1061324363 \pm 0.791208019i).
\end{cases}$

Then, with $\Delta > 0$ and the inequality (13), one can easily obtain that the two points $E_2$ and $E_3$ are of hyperbolic saddle foci type, but the point $E_1$ is not of this type; then there are no homoclinic or heteroclinic orbits of Silnikov type that are doubly asymptotic to the equilibrium $E_1$.

**IV. THE EXISTENCE OF HETEROCLINIC ORBITS IN THE SCHIMIZU-MORIOKA SYSTEM**

In this part, we will investigate using the undetermined coefficient method to prove the existence of heteroclinic orbits of the system (7).

**IV-1. Finding heteroclinic orbits**

From (7), we find that:

$$
\begin{cases}
y = \dot{x}, \\
\dot{y} = \ddot{x}, \\
z = \frac{(x-\lambda \dot{x}-\ddot{x})}{x}, \\
\dot{z} = \frac{(x-\lambda \dot{x}-\ddot{x})x-\ddot{x}(x-\lambda \dot{x}-\ddot{x})}{x^2}.
\end{cases}
$$

(14)

Substituting (14) into the third equation of the system (7) gives

$$
x (\dddot{x} + (\lambda + \alpha) \ddot{x} + \alpha \lambda \dot{x} - \alpha x) - \dot{x} \ddot{x} - \lambda \dot{x}^2 + x^4 = 0.
$$

(15)

If $x(t)$ is found, then $z(t)$ and $y(t)$ will also be determined. Therefore, finding the heteroclinic orbits of system (7) is now reduced to seeking a function $\zeta(t)$ such that $\zeta(t) = x(t)$ satisfies (15) and

$$
\begin{align*}
\zeta(t) \to -\delta & \quad \text{as } t \to +\infty, \\
\zeta(t) \to \delta & \quad \text{as } t \to +\infty, \\
\zeta(t) \to \delta & \quad \text{as } t \to -\infty, \\
\zeta(t) \to -\delta & \quad \text{as } t \to -\infty.
\end{align*}
$$
Without loss of generality, one may stipulate a definite direction as follows: from $E_2$ to $E_3$ corresponds to $t \to +\infty$, while from $E_3$ to $E_2$ corresponds to $t \to -\infty$.

Let
\[ x(t) = \zeta(t) = -\delta + \sum_{k=1}^{\infty} a_k e^{k\beta t}, \quad t > 0 \text{ and } \delta = +\sqrt{\alpha}, \] (16)

where $\beta < 0$ is an undetermined constant, and $a_k (k \geq 1)$ are undetermined coefficients.

Substituting (16) into Eq. (15), we get
\[ \sum_{k=1}^{\infty} (G(\beta kt)a_k e^{\beta kt} = H_1 + H_2 + H_3, \] (17)

where
\begin{align*}
H_1 &= \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} \left[ -(\beta^3 + (\alpha + \lambda)\beta^2 t^2 + (\alpha \lambda \beta^2 + (\beta^2 + \lambda \beta)) (k - i)\beta - 6\delta^2) a_i a_{k-i} e^{\beta k t}, \right. \\
H_2 &= 4\delta \sum_{k=3}^{\infty} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} a_i a_{j-i} a_{k-j} e^{\beta k t}, \\
H_3 &= -\sum_{k=4}^{\infty} \sum_{m=3}^{k-1} \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} a_i a_{j-i} a_{m-j} a_{k-m} e^{\beta k t},
\end{align*}

and
\[ G(\beta k) = -\delta (\beta^3 k^3 + (\alpha + \lambda)\beta^2 k^2 + \alpha \lambda \beta k + 2\alpha). \] (18)

Assume that $a_1 \neq 0$, otherwise one can inductively have $a_k = 0$ for all $k > 1$. Now, comparing the coefficients of $e^{\beta k t} (k \geq 1)$ of the same power terms, we obtain the following results.

For $k = 1$,
\[ -\delta (\beta^3 + (\alpha + \lambda)\beta^2 + \alpha \lambda \beta + 2\alpha) = 0, \] (19)

which is just the characteristic polynomial of the Jacobian of the linearized equation of system (7) evaluated at the equilibrium point $E_2$ or $E_3$. Since (10) has a unique negative root for given parameters, there exist an $\alpha < 0$ such that $G(\alpha) = 0$, and for $k > 1$,
\[ G(\beta k) = -\delta (\beta^3 k^3 + (\alpha + \lambda)\beta^2 k^2 + \alpha \lambda \beta k + 2\alpha) \neq 0, \quad k > 1. \]

So, for $k = 2$,
\[ a_2 = \frac{a_1^2 (H_4)}{G(2\beta)}, \] (20)

where
\[ H_4 = (- (\beta^3 + (\alpha + \lambda)\beta^2 + \alpha \lambda \beta) \beta + (\beta^2 + \lambda \beta) \beta - 6\delta^2) a_1^2. \]
For $k = 3$,
\[
a_3 = \frac{[H_5 + H_6]}{G(3\beta)},
\]
where
\[
H_5 = \sum_{i=1}^{2} \left[ - (\beta^3 i^3 + (\alpha + \lambda)\beta^2 i^2 + (\alpha \lambda i)\beta i) + (\beta^2 i^2 + \lambda i) (3 - i)\beta - 6\delta^2 a_i a_{3-i} \right],
\]
\[
H_6 = 4\delta a_3^2.
\]
Finally, for $k \geq 4$,
\[
a_k = \frac{[H_7 + H_8 + H_9]}{G(\beta k)},
\]
where
\[
H_7 = \sum_{i=1}^{k-1} \left[ - (\beta^3 i^3 + (\alpha + \lambda)\beta^2 i^2 + (\alpha \lambda i)\beta i) + (\beta^2 i^2 + \lambda i) (k - i)\beta - 6\delta^2 a_i a_{k-i} \right],
\]
\[
H_8 = 4\delta \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} a_i a_{j-i} a_{k-j},
\]
\[
H_9 = -\sum_{m=3}^{k-1} \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} a_i a_{j-i} a_{m-j} a_{k-m}.
\]
So $\beta$ is completely determined by $\alpha$ and $\lambda$, and $a_k$ ($k \geq 2$) is completely determined by $\alpha$, $\lambda$, $\beta$.

In fact, the first part of the heteroclinic orbit corresponding to $t > 0$ has been determined. Next, the second part corresponding to $t < 0$ will be constructed. Due to the symmetry of the system (7) around the z-axis, if $((x(t), y(t), z(t))$ is a solution, then $(-x(t), -y(t), z(t))$ is also a solution for the system (7). Thus, for $t < 0$, we have:
\[
x(t) = \delta - \sum_{k=1}^{\infty} a_k e^{-\alpha kt}, \quad \delta = \sqrt{\alpha} \text{ and } t < 0.
\]

Then, we can see that the system (7) has a heteroclinic orbit, see Fig. 2, which connects the equilibrium points $E_3$ and $E_2$, and takes the following form:
\[
x(t) = \zeta(t) = \begin{cases} 
-\delta + \sum_{k=1}^{\infty} a_k e^{\beta kt}, & \text{for } t > 0 \\
0, & \text{for } t = 0 \\
\delta - \sum_{k=1}^{\infty} a_k e^{-\beta kt} & \text{for } t < 0
\end{cases}
\]
From the continuity of the solution, we have
\[ \sum_{k=1}^{\infty} a_k = \delta, \] (24)
which will determine the value of \( a_1 \).

**IV-2. The uniform convergence of the heteroclinic orbits series expansion**

The uniform convergence of the series expansion (16) of the heteroclinic orbit is investigated. For simplicity, we only consider the case in which the system (7) has the special parameter set that generates two-scroll attractors. For other parameter sets, the proof is similar if the heteroclinic orbit exists.

For the chaotic Schimizu-Morioka system, \( \lambda = 0.75 \), \( \alpha = 0.45 \), and \( \delta = 0.6708 \), and the values of \( \beta \) and \( a_k \) can be determined by (19)–(22) and (24) as \( |a_2| = 0.3052615187a_1^2 \), \( |a_3| = 0.07902465643 |a_1|^2 \), \( |a_4| = 0.00251644 |a_1|^2 \); one can inductively prove that \( |a_k| < 10^{-(k+1)} |a_1^k|, (k \geq 4) \). We need to seek \( a_1 \) with \( \sum_{k=1}^{\infty} a_k = -\delta = -0.6708 \). Numerical simulation shows that a “stable” \( a_1 \) indeed exists near \(-0.8390447\) with relative error no greater than 1%. So when \( k \geq 4 \), \( a_k \) is bounded, that is there exists an \( l > 0 \), such that \( |a_k| \leq l, k = 1, 2, \cdots \). Consequently, \( \sum_{k=1}^{\infty} |a_k e^{\alpha kt}| \leq l \sum_{k=1}^{\infty} e^{\alpha kt} \) is convergent on \((0, +\infty)\). So \( -\delta + \sum_{k=1}^{\infty} a_k e^{\alpha kt} \) is convergent on \((0, +\infty)\). Similarly, the convergence of \( \delta - \sum_{k=1}^{\infty} a_k e^{-\alpha kt} \) on \((-\infty, 0)\) can also be proved.
From the above discussion, Theorem 1, and condition (13), we can obtain the following theorem.

**Theorem 3.** If \( c > 0 \), \( \Delta > 0 \), and condition (13) are satisfied, then the system (7) has one Šil’nikov heteroclinic orbit, of which one component has the form (23), and the corresponding chaos is of horseshoe type.

Obviously, the typical parameters \( \lambda = 0.75 \), \( \alpha = 0.45 \) are always satisfied. So there exist heteroclinic orbits of Šil’nikov type, and, as a result, there exists a countable number of Smale horseshoes. Therefore, there exists an invariant set constituting the complex attractor. That is the essence of the geometric structure of the attractor.

V. STRUCTURE OF THE ZHOU SYSTEM

In this section, we will investigate the existence of the homoclinic and heteroclinic orbits in the Zhou system [26]. It analytically demonstrates that the Zhou system has one heteroclinic orbit of Šil’nikov type that connects two nontrivial singular points. The Šil’nikov criterion guarantees that the Zhou system has Smale horseshoes and horseshoe chaos. In addition, there also exists one homoclinic orbit joined to the origin. The uniform convergence of the series expansions of these two types of orbits are proved in this section. It is shown that the heteroclinic and homoclinic orbits together determine the geometric structure of Zhou system.

The Zhou system can be described by the following differential equation:

\[
\begin{align*}
\frac{dx}{dt} &= a(y - x), \\
\frac{dy}{dt} &= bx - xz, \\
\frac{dz}{dt} &= xy + cz,
\end{align*}
\]  

(25)

where \( a, b, \) and \( c \) are real parameters. This system exhibits a chaotic attractor as shown in Fig. 3 when \( a = 10, b = 16, \) and \( c = -1 \). When \( a = -0.7, b = 16, \) and \( c = -1, \) the Zhou system has chaotic attractors as shown in Fig. 4. The Zhou system (25) has three equilibrium points \( E_1 = (0,0,0), E_{2,3} = (\pm \sqrt{-bc}, \pm \sqrt{-bc}, b). \)

Then, the characteristic equation of the system (25) at the the point \((x, y, z)\) is

\[
\lambda^3 + (a - c)\lambda^2 - (ab + ac - yx - az)\lambda + x^2a + xy a + acb - caz = 0.
\]  

(26)

The characteristic equation of the system (25) at \( E_1 \) is \((\lambda - c) (\lambda^2 + a\lambda - ab) = 0. \)

Then, the eigenvalues are \( c \) and \( \frac{-a \pm \sqrt{a^2 + 4ab}}{2}. \)

One can see that the equilibrium \( E_1 = (0,0,0) \) will be a saddle focus iff:

\[a^2 + 4ab < 0 \text{ and } -a < 0.\]

(27)
FIG. 3: Phase portraits of the Zhou system in three-dimensions when $a = 10$, $b = 16$, and $c = -1$.

FIG. 4: Phase portraits of the Zhou system in three-dimensions when $a = -0.7$, $b = 16$, and $c = -1$.

The characteristic polynomial of the Zhou system (25) $E_2 = (\sqrt{-bc}, \sqrt{-bc}, b)$ will be

$$\lambda^3 + (a - c)\lambda^2 - (bc + ac)\lambda - 2abc = 0.$$  \hfill (28)

If $c < 0$, due to Descartes’ rule of signs, the characteristic equation (28) has no positive real root. Thus, it has at least one negative real root.

Let $\lambda = \mu - \frac{(a-c)}{3}$, then Equation (28) becomes

$$\mu^3 + p\mu + q = 0,$$  \hfill (29)

where $p = -(bc + ac) - \frac{(a-c)^2}{3}$, $q = \frac{2(a-c)^3}{27} + \frac{(bc+ac)(a-c)}{3} - 2abc$.

Furthermore, denote

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3;$$
when $\Delta > 0$ by the Cardan formula, Equation (29) has a unique negative real root, $\gamma$, and a conjugate pair of complex roots, $\rho \pm i\omega$, with

$$
\gamma = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}},
$$

$$
\rho = -\frac{1}{2} \left( \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} \right),
$$

$$
\omega = \sqrt[3]{\frac{3}{2} - \sqrt{\frac{3}{2} + \sqrt{\frac{3}{2} - \sqrt{\Delta}}} + \sqrt[3]{\frac{3}{2} - \sqrt{\frac{3}{2} - \sqrt{\Delta}}}}.
$$

Therefore, when $\Delta > 0$, the algebraic equation (28) has the following three roots:

$$
\lambda_1 = -\frac{a-c}{3} + \gamma, \quad \lambda_{2,3} = -\frac{a-c}{3} + \rho \pm i\omega,
$$

respectively, where $\lambda_1 < 0$. To ensure that the real part of the complex conjugate roots is positive, it is further required that

$$
-\sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} > \frac{2(a-c)}{3} > 0.
$$

Note that the characteristic polynomial of the Jacobian of the linearized system of (25), evaluated at the equilibrium point $E_3$, is exactly the same as (28). So, it also has three roots, identical to (30).

**VI. EXISTENCE OF ŠIPLNIKOV-TYPE ORBITS**

**VI-1. The existence of heteroclinic orbits**

In this part, we will investigate the undetermined coefficient method to prove the existence of heteroclinic orbits of the system (25).

From (25), we find that

$$
\begin{align*}
\dot{y} &= \frac{\dot{x}}{a} + \dot{x}, \\
\dot{z} &= \frac{(b\dot{x} - \dot{y})}{x}, \\
x &= \frac{(b\dot{x} - \dot{y})}{\dot{x} - \dot{x}(b\dot{x} - \dot{y})}.
\end{align*}
$$

Substituting (32) into the third equation of system (25) gives

$$
\left(x\left(\ddot{x} + (a-c)\dot{x} - ac\dot{x} + abc\dot{x}\right)\right) - \dot{x}\left(\ddot{x} + a\dot{x} + x^3\right) + ax^4 = 0.
$$

If $x(t)$ is found, then $z(t)$ and $y(t)$ will also be determined. Therefore, finding a heteroclinic orbit of system (25) is now changed to seeking a function $\zeta(t)$ such that $\zeta(t) = x(t)$ satisfies (33) and

$$
\begin{align*}
\zeta(t) &\to -\sqrt{-bc} \quad \text{as } t \to +\infty, \\
\zeta(t) &\to \sqrt{-bc} \quad \text{as } t \to -\infty, \\
\zeta(t) &\to -\sqrt{-bc} \quad \text{as } t \to -\infty, \\
\zeta(t) &\to -\sqrt{-bc} \quad \text{as } t \to -\infty.
\end{align*}
$$
Without loss of generality, one may stipulate a definite direction as follows: from $E_2$ to $E_3$ corresponds to $t \to +\infty$, while from $E_3$ to $E_2$ corresponds to $t \to -\infty$.

Let

$$x(t) = \zeta(t) = -\delta + \sum_{k=1}^{\infty} a_k e^{\alpha kt}, \quad \delta = \sqrt{-bc}, \quad \text{and} \quad t > 0,$$

(34)

where $\alpha < 0$ is an undetermined constant and $a_k (k \geq 1)$ are undetermined coefficients.

Substituting from (34) into Equation (33), we obtain

$$\sum_{k=1}^{\infty} (G(\alpha k) a_k e^{\alpha kt}) = H_1 + H_2 + H_3,$$

(35)

where

$$H_1 = \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} [\alpha^3 i^3 + (a - c) \alpha^2 i^2 - (ac) \alpha i - (\alpha^2 i^2 + a \alpha i) (k - i) \alpha + 6a \delta^2$$

$$- 3\alpha \delta^2 i + abc a_i a_{k-i} e^{\alpha kt},$$

$$H_2 = -\delta \sum_{k=3}^{\infty} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} (4a + 3\alpha (k - j)) a_j a_{k-j-1} a_{k-j} e^{\alpha kt},$$

$$H_3 = \sum_{k=4}^{\infty} \sum_{m=3}^{k-1} \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} (a + \alpha (k - m)) a_i a_j a_{m-j} a_{k-m} e^{\alpha kt},$$

and

$$G(\alpha k) = \delta \left( \alpha^3 k^3 + (a - c) \alpha^2 k^2 - (ac + bc) \alpha k - 2abc \right).$$

Comparing the coefficients of $e^{\alpha kt} (k \geq 1)$ of the same power terms, we obtain the following results.

For $k = 1$,

$$\alpha^3 + (a - c) \alpha^2 - (ac + bc) \alpha - 2abc = 0,$$

(36)

which is just the characteristic polynomial of the Jacobian of the linearized equation of system (25) evaluated at the equilibrium point $E_2$ or $E_3$. So (36) has a unique negative root for the given parameters, there exists an $\alpha < 0$ such that $G(\alpha) = 0$, and for $k > 1$,

$$G(\alpha k) = \delta \left( \alpha^3 k^3 + (a - c) \alpha^2 k^2 - (ac + bc) \alpha k - 2abc \right) \neq 0.$$

(37)

That is

$$G(\alpha k) = \delta \left( \alpha^3 k^3 + (a - c) \alpha^2 k^2 - (ac + bc) \alpha k - 2abc \right) \neq 0, \quad k > 1.$$
So, for $k = 2$,

$$a_2 = \frac{a_1^2 (H_4)}{G(2\alpha)},$$

where

$$H_4 = \left( \alpha^3 + (a - c)\alpha^2 - (ac)\alpha + abc \right) + \left( \alpha^2 + a\alpha - 3\delta \right) \alpha + 6\delta^2.$$

For $k = 3$,

$$a_3 = \frac{[H_5 + H_6]}{G(3\alpha)},$$

where

$$H_5 = \sum_{i=1}^{2} \left[ \alpha^3 i^3 + (a - c)\alpha^2 i^2 - (ac)\alpha i - (\alpha^2 i^2 + a\alpha i) (k - i) \alpha + 6a\delta^2 - 3\alpha\delta^2 i + abc \right] a_i a_{3-i},$$

$$H_6 = -\delta \left( 4a + 3\alpha \right) a_1^3.$$

Finally, for $k \geq 4$,

$$a_k = \frac{[H_7 + H_8 + H_9]}{G(\alpha k)},$$

where

$$H_7 = \sum_{i=1}^{k-1} \left[ \alpha^3 i^3 + (a - c)\alpha^2 i^2 - (ac)\alpha i - (\alpha^2 i^2 + a\alpha i) (k - i) \alpha + 6a\delta^2 - 3\alpha\delta^2 i + abc \right] a_i a_{k-i},$$

$$H_8 = -\delta \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \left( 4a + 3\alpha (k - j) \right) a_i a_{j-i} a_{k-j},$$

and

$$H_9 = \sum_{m=3}^{k-1} \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \left( a + \alpha (k - m) \right) a_i a_{j-i} a_{m-j} a_{k-m}.$$

So $\alpha$ is completely determined by $a, b,$ and $c,$ and $a_k$ ($k \geq 2$) is completely determined by $a, b, c,$ and $\alpha$.

In fact, the first part of the heteroclinic orbit corresponding to $t > 0$ has been determined. Next, its second part corresponding to $t < 0$ will be constructed.

Due to the symmetry of the system, one component of the heteroclinic orbit of (25) has the following form:

$$x(t) = \zeta(t) = \begin{cases} 
-\delta + \sum_{k=1}^{\infty} a_k e^{\alpha k t}, & \text{for } t > 0 \\
0, & \text{for } t = 0 \\
\delta - \sum_{k=1}^{\infty} a_k e^{-\alpha k t}, & \text{for } t < 0.
\end{cases}$$

(41)
VI-2. The uniform convergence of heteroclinic orbits series expansion

The uniform convergence of the heteroclinic orbit series expansion is investigated. For simplicity, we only consider the case in which system (25) has the special parameter set that generates two-scroll attractors. For other parameter sets, the proof is similar if the heteroclinic orbit exists.

When \( a = 10, \ b = 16, \ c = -1 \), and \( \delta = \sqrt{-bc} = 4 \), the values of \( \alpha \) and \( a_k \) can be determined by (36)–(40), and (42), as \( |a_2| = 0.009279499509a_1^2 \), \( |a_3| = 0.0001944176542 |a_1^3| \), and \( |a_4| = 0.0000109295777a_1^4 \); one can inductively prove that \( |a_k| < 10^{-k+1} |a_1^k| \), \( (k \geq 4) \).

We need to seek \( a_1 \) with \( \sum_{k=1}^{\infty} a_k e^{ak t} = \delta \). Numerical simulation shows that a “stable” \( a_1 \) indeed exists near 3.85125 with relative error no greater than 1%. So when \( (k \geq 4) \) \( a_k \) is bounded, that is there exists an \( l > 0 \), such that \( |a_k| \leq l, k = 1, 2, \ldots \).

Consequently, \( \sum_{k=1}^{\infty} |a_k e^{ak t}| \leq l \sum_{k=1}^{\infty} e^{ak t} \) is convergent on \((0, +\infty)\). So \( -\delta + \sum_{k=1}^{\infty} a_k e^{ak t} \) is convergent on \((0, +\infty)\). Similarly, the convergence of \( \delta - \sum_{k=1}^{\infty} a_k e^{-ak t} \) on \((-\infty, 0)\) can also be proved.

From the above discussion, Theorem 1, and condition (31), we can obtain the following theorem.

**Theorem 4.** If \( \Delta > 0, \ c < 0 \), and condition (31) are satisfied, then the system (25) has one
Si’lnikov heteroclinic orbit of which one component has the form (41), and the corresponding chaos is of horseshoe type.

VI-3. The existence of homoclinic orbits

Recall that a homoclinic orbit joining the origin of the Zhou system (25) implies that such an orbit is doubly asymptotic with respect to time $t$ to the origin. As in the previous subsection, let $x(t) = \varphi(t)$, which tends to $y(t) = \overset{\cdot}{\varphi}(t)$, $z(t) = (b\varphi(t) - \overset{\cdot}{y})\varphi(t) - \overset{\cdot}{\varphi}(t) (b\varphi(t) - \overset{\cdot}{y})$, (43)

and $\varphi(t)$ satisfies

$\varphi(t) (\overset{\cdot}{\varphi}(t) + (a - c)\overset{\cdot}{\varphi}(t) - ac\varphi(t) + abc\varphi(t)) - \overset{\cdot}{\varphi}(t) (\overset{\cdot}{\varphi}(t) + a\overset{\cdot}{\varphi}(t) + \varphi(t)^3) + c\varphi(t)^4 = 0$. (44)

Note, at the origin, it is required that $\varphi(t) \to 0$ as $t \to \pm \infty$. Assume that for $t > 0$

$\varphi(t) = \sum_{k=1}^{\infty} d_k e^{\beta k t}$,

(45)

where $\beta < 0$ is an undetermined constant and $(d_k \geq 1)$ are undetermined coefficients. Then similar to the discussion in the case of the heteroclinic orbit, we have the following.

Given all the similarities described above, it is important to point out that the exponent component $\beta$ in the homoclinic orbit is completely different from that in the heteroclinic orbit. In addition,

$e \sum_{k=2}^{\infty} \sum_{i=1}^{k-1} (\beta^2 i^2 + a\beta i - ab) d_i d_{k-i} e^{\beta k t}$

$= a \sum_{k=4}^{\infty} \sum_{m=3}^{k-1} \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} d_i d_{j-i} d_{m-j} d_{k-m} e^{\beta k t}$

$- \sum_{k=4}^{\infty} \sum_{m=3}^{k-1} \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} d_i d_{j-i} d_{m-j} (k - m) \beta d_{k-m} e^{\beta k t}$. (46)

Then, comparing the coefficients of $e^{\beta k t} (k \geq 2)$ of the same power terms, we obtain the following results.

For $k = 2$,

$\beta^2 + a\beta - ab = 0$,

which is just the characteristic equation of the linearized equation (25) evaluated at the point $E_0$. Then we can obtain

$\beta = \frac{-a \pm \sqrt{a^2 + 4ab}}{2}$. 
For $k = 3$,

$$5\beta^2 + 3a\beta - 2ab \cdot cd_1d_2 = 0.$$  \hfill (48)

For $k = 4$,

$$c \left[ (10\beta^2 + 4a\beta - 2ab)d_1d_3 + (4\beta^2 + 2a\beta - ab)d_2^2 \right] + (\beta - a)d_1^4 = 0.$$  \hfill (49)

For $k = 5$,

$$c(17\beta^2 + 5a\beta - 2ab)d_1d_4 = 0.$$  \hfill (50)

From (43), one can get that $d_1 = 0$ or $d_2 = 0$. If $d_1 = 0$, one can inductively obtain $d_k = 0$ for all $k \geq 2$ by (43). So $d_2 = 0$.

From (48) and (49), we obtain

$$d_3 = \frac{(a - \beta)d_1^3}{c(10\beta^2 + 4a\beta - 2ab)} = G_3(\beta)d_1^3,$$  \hfill (51)

and, $d_4 = 0$, respectively. So, a more general result can be inductively obtained from $d_2 = 0$ and (46), which is:

$$d_{2k} = 0, \quad d_{2k+1} = G_{2k+1}(\beta)d_1^{2k+1}, \quad k \geq 1.$$  \hfill (52)

Finally, due to the symmetry of the Zhou system (25), one component of the homoclinic orbit of system (25) has the following form:

$$x(t) = \varphi(t) = \begin{cases} 
\sum_{k=1}^{\infty} d_{2k+1}e^{\beta kt}, & \text{for } t > 0 \\
0, & \text{for } t = 0 \\
\sum_{k=1}^{\infty} d_{2k+1}e^{-\beta kt}, & \text{for } t < 0.
\end{cases}$$  \hfill (53)

The convergence of the series expansions of the homoclinic orbits can be proved, similar to the discussion in Section VI.2, and the proof is omitted here.

From the above discussion, Theorem 2, and condition (27), we can obtain the following result.

**Theorem 5.** If $a^2 + 4ab < 0$ and $-ac < 0$ are satisfied, then the system (25) has one Ši’lnikov homoclinic orbit of which one component has the form (53), and the corresponding chaos is of horseshoe type.

The homoclinic orbits and heteroclinic orbits cannot exist at the same time for the Zhou system. In the case of heteroclinic orbits, we take $a = 10$, $b = 16$, and $c = -1$, then the equilibria and their eigenvalues are given by

$$\begin{array}{l}
E_2 = (4, 4, 16), \quad (\lambda_1, \lambda_2, \lambda_3) = (-11.22372541, 0.1118627027 \pm 5.33840044I), \\
E_3 = (-4, -4, 16), \quad (\lambda_1, \lambda_2, \lambda_3) = (-11.22372541, 0.1118627027 \pm 5.33840044I).
\end{array}$$
In the case of homoclinic orbits the parameter values \( a = -0.7, b = 10, \) and \( c = -1, \) then the equilibria and their eigenvalues are given by
\[
E_1 = (0, 0, 0), \text{ then } (\lambda_1, \lambda_2, \lambda_3) = (-1, 0.35 \pm 3.328278481i).
\]

Then, it is clear that all \( E_i, i = 1, 2, 3 \) of the Zhou system are of hyperbolic saddle foci type. Which gives the possibility to get homoclinic or heteroclinic orbits for the Zhou system, and that is what we proved in this section.

VII. CONCLUSIONS

The existence of Ši’lnikov chaos in both the Schimizu-Morioka system and the Zhou system has been investigated. By using the undetermined coefficient method, we have found the heteroclinic orbit of the Schimizu-Morioka system, with an explicit and uniformly convergent algebraic expression; on the other hand, the existence of two types of orbits in the Zhou system, i.e., heteroclinic and homoclinic orbits with the explicit and uniformly convergent algebraic expressions, has been proved. By the Ši’lnikov criterion, both of the two systems have Smale horseshoe chaos.

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