The Super Kaup-Newell Soliton Hierarchy and its Super-Hamiltonian Structure

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Based on the super-Lie algebra $B(0,1)$, the super-Kaup-Newell soliton hierarchy is derived. The super-Hamiltonian structure is constructed by using the super-trace identity. An infinite set of commuting conserved functionals and commuting symmetries of the resulting super hierarchies are given.

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I. INTRODUCTION

Since the discovery of the integrability of the Korteweg-de Vries equation, there has been a rapid growth of what is nowadays called the theory of integrable systems. This theory is still in active development, mainly because it is an effective tool for describing and explaining nonlinear phenomena, such as nonlinear optics, super-conductivity, plasma physics, magnetic fluid, etc. A powerful tool, i.e., the trace identity, for constructing the Hamiltonian structures of integrable systems presented by Tu [1] has been extensively applied to many nonlinear integrable soliton equations [2–4]. A key feature of an integrable equation is the fact that it can be expressed as the compatibility condition of two suitable linear equations, usually referred to as a Lax pair.

In the last few years, the super-integrable systems have received much attention, especially in the explorations of the application to the supersymmetric conformal field theories and string theories [5]. This has resulted in the supersymmetrization of existing integrable equations and the extension of the methods involved in the study of integrable hierarchies to the super-integrable framework [6, 7]. These super-integrable systems are shown to have many common features [8–11]. Furthermore, it is a common belief that they also possess Lax representations and bi-Hamiltonian structures [12, 13] that define the dynamical flows on the corresponding Poisson supermanifolds. In order to look for the super-Hamiltonian structure of the corresponding super-integrable system, the standard trace identity [1] is generalized to a super-trace identity, by use of which the super-Hamiltonian structures of some super systems were successfully established [14, 15].

In contrast with the general soliton equations, which are based on the Killing form on a semi-simple Lie algebra, the super-trace identity is associated with a commutative
super-algebra $\mathcal{A}$ defined over $\mathbb{R}$ or $\mathbb{C}$ with the non-degenerate Killing form. Let $G$ be a matrix loop super-algebra over $\mathcal{A}$ with a non-degenerate Killing form. For an operator $J = (J_{ij})_{q \times q}$ from $\mathcal{A}^q$ to $\mathcal{A}^q$, the corresponding bracket is defined by

$$\{H_1, H_2\} = \int \sum_{i,j=1}^q (-1)^{p(i)p(j)p(H_2)} \left( J_{ij} \frac{\delta H_2}{\delta u_j} \right) \frac{\delta H_1}{\delta u_i} dx,$$

(1.1)

where $H_1, H_2$ are two functionals, and, $p(i) = p(u_i)$ and $p(H_2)$ are the degrees of $u_i$ and $H_2$ (either 0 or 1). An operator $J$ is called a super-Hamiltonian operator if the corresponding bracket (1.1) is a super-Lie algebra, i.e., it is superskewsymmetric,

$$\{H_1, H_2\} = -(-1)^{p(H_1)p(H_2)} \{H_2, H_1\},$$

and satisfies the super-Jacobi identity

$$(-1)^{p(H_1)p(H_2)} \{H_1, \{H_2, H_3\}\} + \text{cyclic} (H_1, H_2, H_3) = 0,$$

(1.2)

$H_i, (i = 1, 2, 3)$ are pure in the $\mathbb{Z}_2$ grading. An evolution equation,

$$u_t = \mathcal{K}(u) = \mathcal{K}(u, u_x, \ldots, \frac{\partial^l u}{\partial x^l}),$$

is called a super-Hamiltonian system [14, 15], if there is a super-Hamiltonian operator $\mathcal{J}$ and function $\mathcal{H}$ such that

$$u_t = \mathcal{K}(u) = \mathcal{J} \frac{\delta \mathcal{H}}{\delta u}.$$

Consequently, the evolution equation is said to have a super-Hamiltonian structure.

II. SUPER-INTEGRABLE KAUP-NEWELL SOLITON HIERARCHY

This section is devoted to introducing super-integrable equations related to the classical Kaup-Newell soliton hierarchy. Set $B(0,1)$ to be a super-Lie algebra given by

$$B(0,1) = \text{span}\{E_i, i = 0, 1, 2, 3, 4\},$$

with

$$E_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
That is

\[ B(0, 1) = \left\{ g \mid g = g_0 E_0 + g_1 E_1 + g_2 E_2 + g_3 E_3 + g_4 E_4 = \begin{pmatrix} g_0 & g_1 & g_3 \\ g_2 & -g_0 & g_4 \\ g_4 & -g_3 & 0 \end{pmatrix}, g_i \in A, 0 \leq i \leq 4 \right\}, \]

where \( A \) is a commutative super-algebra over \( R \) or \( C \), \( g_0, g_1, g_2 \) are even, and \( g_3, g_4 \) are odd [14–17]. The commutation and anti-commutation relations are defined by [14–17]

\[ [a, b] = ab - (-1)^{p(a)p(b)} ba, \quad a, b \in B(0, 1) \]

i.e.,

\[ [E_0, E_1] = 2E_1, \quad [E_0, E_2] = -2E_2, \quad [E_1, E_2] = E_0, \quad [E_0, E_3] = E_3, \]

\[ [E_0, E_4] = -E_4, \quad [E_1, E_3] = 0, \quad [E_1, E_4] = E_3, \quad [E_2, E_3] = E_4, \]

\[ [E_2, E_4] = 0, \quad [E_3, E_3]^+ = -2E_1, \quad [E_3, E_4]^+ = E_0, \quad [E_4, E_4]^+ = 2E_2. \]

Then we have the super loop Lie algebra

\[ G = \tilde{B}(0, 1) = B(0, 1) \otimes C[\lambda, \lambda^{-1}] \quad \text{or} \quad B(0, 1) \otimes R[\lambda, \lambda^{-1}]. \]

Now, let us consider the spectral problem associated with \( \tilde{B}(0, 1) \) as follows:

\[ \phi_x = U \phi, \quad U = E_0(1) + rE_1(0) + sE_2(1) + \alpha E_3(0) + \beta E_4(1), \quad u = \begin{pmatrix} r \\ s \\ \alpha \\ \beta \end{pmatrix}, \quad (2.1) \]

where \( \lambda \) is the spectral parameter with \( \lambda_0 = 0, r, s \) are commuting variables, and \( \alpha, \beta \) are anti-commuting variables [14, 15]. The above super-spectral problem reduces to a standard Kaup-Newell spectral problem [18] if we set \( \alpha = \beta = 0 \).

The stationary zero curvature equation

\[ V_x = [U, V], \quad (2.2) \]

with

\[ V = AE_0(0) + BE_1(0) + CE_2(1) + \rho E_3(0) + \sigma E_4(1), \]
yields

\[
A_x = \lambda rc + \lambda \alpha \sigma - \lambda s B + \lambda \beta \rho, \\
B_x = 2 \lambda B - 2r A - 2 \alpha \rho, \\
C_x = -2 \lambda C + 2s A + 2 \lambda \beta \sigma, \\
\rho_x = \lambda \rho + r \lambda \sigma - \alpha A - \lambda \beta B, \\
\sigma_x = -\lambda \sigma + s \rho - \alpha C + \beta A.
\] (2.3)

The substitution of the selection

\[
A = \sum_{i=0}^{\infty} A_i \lambda^{-i}, \quad B = \sum_{i=0}^{\infty} B_i \lambda^{-i}, \quad C = \sum_{i=0}^{\infty} C_i \lambda^{-i}, \quad \rho = \sum_{i=0}^{\infty} \rho_i \lambda^{-i}, \quad \sigma = \sum_{i=0}^{\infty} \sigma_i \lambda^{-i},
\]

into (2.3) leads to the initial relations

\[
A_0 = 1, \quad B_0 = C_0 = \rho_0 = \sigma_0 = 0,
\]

and the recursion relations

\[
A_{mx} = r C_{m+1} + \alpha \sigma_{m+1} - s B_{m+1} + \beta \rho_{m+1}, \\
B_{mx} = 2 B_{m+1} - 2r A_m - 2 \alpha \rho_m, \\
C_{mx} = -2 C_{m+1} + 2s A_m + 2 \beta \sigma_{m+1}, \quad m \geq 0. \quad (2.4)
\]

\[
\rho_{mx} = \rho_{m+1} + r \sigma_{m+1} - \alpha A_m - \beta B_{m+1}, \\
\sigma_{mx} = -\sigma_{m+1} + s \rho_m - \alpha C_m + \beta A_m.
\]

Similarly, by selecting the zero-constants in the integration operation, we have

\[
A_1 = -\frac{r s}{2} - \alpha \beta, \quad B_1 = r, \quad C_1 = s, \quad \rho_1 = \alpha, \quad \sigma_1 = \beta,
\]

\[
A_2 = -\frac{1}{4} (r s x - r x s) + \alpha \beta x - \alpha x \beta + r s \alpha \beta + \frac{3}{8} r^2 s^2 + 2r \beta \beta x,
\]

\[
B_2 = \frac{1}{2} r x - \frac{1}{2} r^2 s - r \alpha \beta, \quad C_2 = -\frac{1}{2} s x - \frac{1}{2} r s^2 - s \alpha \beta - \beta \beta x,
\]

\[
\rho_2 = \alpha x + r \beta x - \frac{1}{2} r s \alpha + \frac{1}{2} r x \beta, \quad \sigma_2 = -\beta x - \frac{1}{2} r x \beta, \quad \ldots.
\]

Now we set

\[
(\lambda^m V)_+ = \sum_{i=0}^{m} \begin{pmatrix} A_i \lambda^{m-i} & B_i \lambda^{m-i} & \rho_i \lambda^{m-i} \\ C_i \lambda^{m-i+1} & -A_i \lambda^{m-i} & \sigma_i \lambda^{m-i+1} \\ \sigma_i \lambda^{m-i+1} & -\rho_i \lambda^{m-i} & 0 \end{pmatrix}.
\] (2.5)
It is not difficult to find

\[(\lambda^mV)_{+x} - [U, (\lambda^mV)_+] \]

\[
\begin{pmatrix}
A_{mx} & 2B_{m+1} & \rho_{m+1} + r\sigma_{m+1} - \beta B_{m+1} \\
2\lambda(\beta\sigma_{m+1} - 2C_{m+1}) & -A_{mx} & -\lambda\sigma_{m+1} - \rho_{m+1} + r\sigma_{m+1} - \beta B_{m+1} \\
-\lambda\sigma_{m+1} & -[\rho_{m+1} + r\sigma_{m+1} - \beta B_{m+1}] & 0
\end{pmatrix}.
\]

So we introduce the modification as follows:

\[\Delta_m = \begin{pmatrix}
-A_m & 0 & 0 \\
0 & A_m & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

and define

\[V^{(m)} = (\lambda^mV)_+ + \Delta_m, \quad m \geq 0.\]

Through a direct calculation, we have

\[V_x^{(m)} - [U, V^{(m)}] = \begin{pmatrix}
0 & B_{mx} + 2\alpha\rho_m & \rho_{mx} \\
\lambda(\sigma_{mx} - s\rho_m + \alpha C_m) & 0 & \lambda(\sigma_{mx} - s\rho_m + \alpha C_m) \\
\lambda(\sigma_{mx} - s\rho_m + \alpha C_m) & -\rho_{mx} & 0
\end{pmatrix},
\]

which is consistent with \(U_{tm}\). Then we introduce the following auxiliary spectral problem:

\[\phi_{tm} = V^{(m)}\phi, \quad m \geq 0. \tag{2.6}\]

The compatibility conditions of (2.1) and (2.6), i.e., the zero-curvature equations

\[U_{tm} = V_x^{(m)} - [U, V^{(m)}],\]

give rise to the following hierarchy of super-KN equations:

\[u_{tm} = K_m \begin{pmatrix}
r \\
s \\
\alpha \\
\beta
\end{pmatrix}_{tm} = \begin{pmatrix}
B_{mx} + 2\alpha\rho_m \\
C_{mx} \\
\rho_{mx} \\
\sigma_{mx} - s\rho_m + \alpha C_m
\end{pmatrix}, \quad m \geq 0. \tag{2.7}\]

When \(m = 2\) the resulting system reduces to

\[\begin{align*}
q_{t_2} &= \frac{1}{2}r_{xx} - \frac{1}{2}(r^2 s)_x + r(\alpha\beta_x - \alpha_x\beta) + 2\alpha\alpha_x, \\
r_{t_2} &= -\frac{1}{2}s_{xx} - \frac{1}{2}(rs^2)_x - (s\alpha\beta)_x - (\beta\beta_x)_x, \\
\alpha_{t_2} &= \alpha_{xx} + \frac{1}{2}(r_x\beta)_x + (r\beta_x)_x - \frac{1}{2}(r\alpha\beta)_x, \\
\beta_{t_2} &= -\beta_{xx} - \frac{1}{2}(rs\beta)_x.
\end{align*} \tag{2.8}\]

When \(\alpha = \beta = 0\) the system (2.7) reduces to the classical Kaup-Newell equation [18].
III. SUPER-HAMILTONIAN STRUCTURE OF THE SYSTEM (2.7)

Given a spectral matrix $U \in G$, we define $\text{rank}(U) = \text{rank}(\frac{\partial}{\partial x}) = \text{const}$. Assume that if two solutions $V_1, V_2 \in G$ of the stationary zero curvature equation,

$$V_x = [U, V],$$

possess the same rank, then they are $A$ linearly dependent on each other, i.e.,

$$V_1 = \gamma V_2, \quad \gamma = \text{const}.$$

From [14] we have the following two theorems.

**Theorem 1** (The super-trace identity) Let $U = U(u, \lambda) \in G$ be homogeneous in rank. Assume that the stationary zero curvature equation has a unique solution $V \in G$ of a fixed rank up to a constant multiplier. Then, there is a constant $\gamma$ such that

$$\frac{\delta}{\delta u} \int \text{str}(\text{ad}_V \text{ad}_{\partial U/\partial \lambda}) dx = \left( \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \right) \text{str}(\text{ad}_V \text{ad}_{\partial U/\partial u}).$$

(3.1)

holds for any solution $V \in G$ of stationary zero curvature equation, being homogeneous in rank.

**Theorem 2** Let $V$ be a solution to the stationary zero curvature equation. The constant in the super-trace identity is given by

$$\gamma = -\frac{\lambda}{2} \frac{d}{dx} \ln |\text{str}(\text{ad}_V \text{ad}_V)|,$$

if $\text{str}(\text{ad}_V \text{ad}_V) \neq 0$.

In what follows, we would like to construct the super-Hamiltonian structure for the super-integrable system (2.7). To this end, we apply the super-trace identity (3.1). The substitution of the selection

$$\text{str}(\text{ad}_V \text{ad}_{\partial U/\partial \lambda}) = 3\text{str}(V, U_\lambda) = 3(2A + sB + 2\beta \rho),$$

$$\text{str}(\text{ad}_V \text{ad}_{\partial U/\partial r}) = 3\lambda C, \quad \text{str}(\text{ad}_V \text{ad}_{\partial U/\partial s}) = 3\lambda B,$$

$$\text{str}(\text{ad}_V \text{ad}_{\partial U/\partial \alpha}) = 3(-2\lambda \sigma), \quad \text{str}(\text{ad}_V \text{ad}_{\partial U/\partial \beta}) = 3(2\lambda \rho),$$

into (3.1) gives rise to

$$\frac{\delta}{\delta u} \int (2A + sB + 2\rho \beta) dx = \left( \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \right) \begin{pmatrix} \lambda C \\ \lambda B \\ -2\lambda \sigma \\ 2\lambda \rho \end{pmatrix}.$$

Comparing the coefficients of $\lambda^{-m-1}$ yields
\[
\frac{\delta}{\delta u} \int \left( 2A_{m+1} + sB_{m+1} + 2\rho_{m+1}\beta \right) dx = (\epsilon - m) \begin{pmatrix} C_{m+1} \\ B_{m+1} \\ -2\sigma_{m+1} \\ 2\rho_{m+1} \end{pmatrix}.
\]

It is easy to verify that \( \gamma = 0 \) if we simply set \( m = 0 \). That is

\[
\frac{\delta}{\delta u} \int \frac{\left( 2A_{m+1} + sB_{m+1} + 2\rho_{m+1}\beta \right)}{m} dx = \begin{pmatrix} C_{m+1} \\ B_{m+1} \\ -2\sigma_{m+1} \\ 2\rho_{m+1} \end{pmatrix}, \quad m \geq 1.
\]

Then, we have the super-Hamiltonian structure of the super-KN hierarchy as follows:

\[
u_{tm} = K_m = J \frac{\delta H_m}{\delta u}, \quad m > 1,
\]

in which the super-Hamiltonian operator \( J \) and the super-Hamiltonian functionals \( H_m \) are given by

\[
J = \begin{pmatrix} 0 & \partial & \alpha \\ \partial & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \partial \\ \alpha & -\frac{1}{2} \partial & -\frac{1}{2} s \end{pmatrix}, \quad H_m = \int \frac{2A_m + sB_m + 2\rho_m\beta}{1 - m} dx, \quad m > 1.
\]

If we set

\[
\frac{\delta H_{m+1}}{\delta u} = L \frac{\delta H_m}{\delta u},
\]

from the recursion relation (2.4) we get the hereditary recursion operator

\[
L = \begin{pmatrix} -\frac{1}{2} (\partial + s\partial^{-1}r\partial) + \alpha \beta & -\frac{1}{2} s\partial^{-1}s\partial & \frac{1}{2} (s\partial^{-1}\alpha\partial + \beta\partial) & \frac{1}{2} (s\partial^{-1}\beta\partial + s\beta) \\ -\frac{1}{2} r\partial^{-1}s\partial & 2\alpha + \beta\partial^{-1}r\partial & -\frac{1}{2} r\partial^{-1}\partial & \frac{1}{2} (\alpha + r\partial^{-1}\beta\partial) \\ 2\alpha - \alpha\partial^{-1}r\partial & -\beta \partial - \alpha\partial^{-1}s\partial & -\partial - \beta\partial^{-1}\alpha\partial & -s - \beta\partial^{-1}\beta\partial \\ 2r\alpha - \alpha\partial^{-1}\partial \partial & -\beta \partial - \alpha\partial^{-1}s\partial & \alpha\partial^{-1}\partial - r\partial - rs - \alpha\beta + \alpha\partial^{-1}\beta\partial \end{pmatrix}.
\]

Then the super-KN soliton hierarchy (2.7) has the following super-bi-Hamiltonian structure:

\[
u_{tm} = K_m = J\frac{\delta H_m}{\delta u} = M \frac{\delta H_{m-1}}{\delta u}, \quad m \geq 1,
\]

where the second compatible operator reads

\[
M = J L = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}.
\]
with
\[ M_{11} = -\frac{1}{2} \partial r \partial^{-1} r \partial, \quad M_{12} = \frac{1}{2} \partial (\partial - r \partial^{-1} s \partial) - \alpha \beta \partial, \]
\[ M_{13} = \frac{1}{2} \partial r \partial^{-1} \alpha \partial - r \alpha \partial, \quad M_{14} = \frac{1}{2} \partial (\alpha + r \partial^{-1} \beta \partial) + \alpha \partial - rs \alpha, \]
\[ M_{21} = -\frac{1}{2} \partial (\partial + s \partial^{-1} r \partial) + \partial \alpha \beta, \quad M_{22} = -\frac{1}{2} \partial s \partial^{-1} s \partial, \]
\[ M_{23} = \frac{1}{2} \partial (\beta \partial + s \partial^{-1} \alpha \partial), \quad M_{24} = \frac{1}{2} \partial (s \partial^{-1} \beta \partial + s \beta), \]
\[ M_{31} = -\frac{1}{2} \partial (\alpha \partial^{-1} r \partial - 2r \alpha), \quad M_{32} = -\frac{1}{2} \partial (\beta \partial + \alpha \partial^{-1} s \partial), \]
\[ M_{33} = \frac{1}{2} \partial (\alpha \partial^{-1} \alpha \partial - r \partial), \quad M_{34} = \frac{1}{2} \partial (\alpha \partial^{-1} \beta \partial + \partial - rs - \alpha \beta), \]
\[ M_{41} = -\frac{1}{2} (\alpha \partial + \beta \partial^{-1} r \partial - \partial \alpha - rs \alpha), \quad M_{42} = \frac{1}{2} (s \beta - \partial \beta \partial^{-1} s \partial), \]
\[ M_{43} = \frac{1}{2} (rs + \alpha \beta + \partial - \partial \beta \partial^{-1} \alpha), \quad M_{44} = \frac{1}{2} (\partial s - s \partial) + \frac{1}{2} \partial \beta \partial^{-1} \beta \partial + \frac{1}{2} rs + sa \beta. \]

**Theorem 3.** The super-Hamiltonian functions \( \{H_m\}_{m \geq 0} \) defined by (3.2) form an infinite set of commuting conserved quantities of the hierarchy (2.7), and \( \{H_m\}_{m \geq 0} \) are in involution in pairs with respect to the super-Poisson bracket (1.1). The hierarchy (2.7) possesses infinitely many commuting symmetries \( \{K_m\}_{m \geq 0} \). So the system (2.7) is a super-integrable Hamiltonian system.

**IV. CONCLUSIONS**

Based on super-Lie algebra \( B(0, 1) \), a hierarchy of super-integrable systems is deduced. It reduces to the standard KN soliton hierarchy if we take the odd potentials as zero. The super-Hamiltonian structure of the super-KN system is constructed by using the super-trace identity.

In contrast with the continuous soliton equations, the super-extensions of discrete soliton equations have been less considered. There would be significance in studying the discrete cases of super-soliton hierarchies. Further, how to construct the integrable and super-integrable coupling system as well as the related Hamiltonian structures would be an interesting topic [20]. These problems shall be discussed on another occasion.

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