Solitary Wave Solutions Having Two Wave Modes of KdV-Type and KdV-Burgers-Type

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Solitary wave solutions having two wave modes of KdV-type with 3rd-order and 5th-order dispersion and KdV-Burgers-type are given.

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I. Introduction

It is well known that the one dimensional propagation of an isolated wave mode in nonlinear dispersive systems which exhibit several different modes is governed by the KdV equation [1]. However, in comparison with the isolated wave mode case, there seems to be relatively less work for two wave modes. The coupled KdV equation, proposed as a possible mode describing the interactions of two long waves with different dispersion relations, was discussed in [2,3]. The solitary solution for two wave modes

\[ L_1 u \equiv u_{tt} + (c_1 + c_2)u_{xt} + c_1c_2 u_{xx} + \left[ (\alpha_1 + \alpha_2) \frac{\partial}{\partial t} + (\alpha_1c_2 + \alpha_2c_1) \frac{\partial}{\partial x} \right] u_x - \left[ (\beta_1 + \beta_2) \frac{\partial}{\partial t} + (\beta_1c_2 + \beta_2c_1) \frac{\partial}{\partial x} \right] u_{xxx} = 0, \]  

was also obtained [4]. In the present paper, we would like to consider two wave modes of KdV-type with 3rd-order dispersion,

\[ L_p u \equiv u_{tt} + (c_1 + c_2)u_{xt} + c_1c_2 u_{xx} + \left[ (\alpha_1 + \alpha_2) \frac{\partial}{\partial t} + (\alpha_1c_2 + \alpha_2c_1) \frac{\partial}{\partial x} \right] u_x - \left[ (\beta_1 + \beta_2) \frac{\partial}{\partial t} + (\beta_1c_2 + \beta_2c_1) \frac{\partial}{\partial x} \right] u_{xxx} = 0, \]  

and two wave modes of KdV-Burgers-type,

\[ L_p u + \left[ (\gamma_1 + \gamma_2) \frac{\partial}{\partial t} + (\gamma_1c_2 + \gamma_2c_1) \frac{\partial}{\partial x} \right] u_{xxxx} = 0. \]
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\[ L_p u + \left[ \left( \sigma_1 + \sigma_2 \right) \frac{\partial}{\partial t} + \left( \sigma_1 c_2 + \sigma_2 c_1 \right) \frac{\partial}{\partial x} u_{xx} \right] = 0, \tag{4} \]

where \( u(x,t) \) and coefficients \( c_i, \alpha_i, \beta_i, \gamma_i \) and \( \sigma_i \) are, respectively, the field function, the phase velocities, the parameters of nonlinearity, dispersion and dissipation. Eqs. (2),(3), and (4) are assumed to govern propagation in the same direction of the two wave modes with the same dispersion relations but with different phase velocities, nonlinearity and dispersion parameters. The Boussinesq equation, describing propagation and interaction of the surface waves moving in different directions in a shallow fluid, is a typical example of this kind of equation. Eqs. (2), (3) and (4) contain several equations of physical interest, for instance, the modes (2-4) are, respectively, reduced to the generalized KdV-type equation

\[ L_p u \equiv u_t + c_i u_x + \alpha_i u^2 u_x + \beta_i u_{xxx} = 0, \tag{5} \]

an integrable model for long wave propagation in nonlinear media with dispersion, the 5th-order general KdV-type equation

\[ L_p u + \gamma_i u_{xxxx} = 0, \tag{6} \]

and the KdV-Burgers-type equation

\[ L_p u + \sigma_i u_{xx} = 0, \tag{7} \]

in the absence of the other wave. Eq. (6) with \( p=1 \) is a 5th-order KdV equation which has applications in fluid mechanics, plasma physics etc. [5-7]. Solitary wave solutions to Eqs. (5) and (6) and their stability have been discussed [8-10]. Eq. (7) with \( p=1 \) has been derived as an equation which describes the propagation of undular bores in shallow water [11,12] and weakly nonlinear plasma wave with certain dissipative effects [13-15]. In this paper, we focus on the solitary wave solutions and some of their properties.

II. Solitary wave solutions to equations (2)-(4)

Introduce the transformations

\[ \xi = (\beta_1 + \beta_2)^{-1/2}(z-\cos \omega t), \quad T = (\beta_1 + \beta_2)^{-1/2} \omega t, \quad c_0 = (c_1 + c_2)/2, U = (a_1 + a_2)^{1/p} u \]

then Eqs. (2),(3), and (4) reduce to, respectively,

\[ F(U) \equiv U_{TT} - \alpha s U_{\xi \xi} + \left( \frac{\partial}{\partial T} + \alpha s \frac{\partial}{\partial \xi} \right) U^p U_t + \left( \frac{\partial}{\partial T} + \beta s \frac{\partial}{\partial \xi} \right) U_{\xi \xi \xi \xi} = 0, \tag{8} \]

\[ F(U) + \left( \frac{\partial}{\partial T} + \gamma s \frac{\partial}{\partial \xi} \right) U_{\xi \xi \xi \xi} = 0, \tag{9} \]

and
\[
F(U) + \alpha \left( \frac{\partial}{\partial T} + \sigma \frac{\partial}{\partial \xi} \right) U_{\xi \xi} = 0, \tag{10}
\]

where

\[
s = (c_1 - c_2)/2, \quad \alpha = (\alpha_2 - \alpha_1)/(c_2 + \alpha_1), \quad \sigma = (\sigma_2 - \sigma_1)/(\sigma_2 + \sigma_1), \quad \beta = (\beta_2 - \beta_1)/c_2, \quad \gamma = (\gamma_2 - \gamma_1)/(\gamma_2 + \gamma_1), \quad \delta = (\sigma_2 \delta + \sigma_1)/(\beta_2 + \beta_1)^{-1/2}.
\]

We consider the traveling wave solutions

\[
U(\xi, T) = V(\eta), \quad \eta = k \xi - \omega T + \eta_0,
\]

where \(k, \omega, \eta_0\) are constants to be determined. It follows from Eqs. (8-10) that

\[
G(V) = (\omega^2 - \delta^2 k^2) V + \left( -\omega k + \alpha \delta k^2 \right) \frac{V^{p+1}}{p+1} + (-\omega k^3 + \beta \delta k^4) V'' = K_1, \tag{11}
\]

\[
G(V) + (-\omega k^2 + \gamma \delta k^3) V''' = K_2, \tag{12}
\]

\[
G(V) + \alpha (-\omega k^2 + \sigma \delta k^3) V' = K_3, \tag{13}
\]

with \(K_i (i = 1, 2, 3)\) being integration constants. It should be remarked that Eqs. (11-13) are different from the equations derived from the travelling wave solution of Eqs. (5-7).

Further, suppose travelling wave solutions to Eqs. (8), (9), and (10) are of the particular forms, respectively,

\[
V(\eta) = A \text{sech}^{2/p} \eta + B \delta p_1, \tag{14}
\]

\[
V(\eta) = A \text{sech}^{2/p} \eta, \tag{15}
\]

\[
V(\eta) = \frac{A_2}{(1 + \exp(2\eta))^{2/p}} + B \delta p_1, \tag{16}
\]

where \(\delta p_1\) is the Kronecker symbol. Substituting Eqs. (14), (15) and (16) into Eqs. (11), (12) and (13), respectively, we obtain the following equations

\[
\omega^2 - \delta^2 k^2 + B \delta p_1 (-\omega k + \alpha \delta k^2) + 4(-\omega k^3 + \beta \delta k^4)/p^2 = 0, \tag{17.1}
\]

\[
A^p = \frac{(4 + 2p)(p+1)(-\omega k^3 + \beta \delta k^4)}{p^2(\omega k - \omega)}; \tag{17.2}
\]

\[
\omega^2 - \delta^2 k^2 - 16(\beta k^4 - \omega k^3)/p^2 + 256k^6/\omega^4 = 0, \tag{18.1}
\]

\[
A^2 = \frac{(4 + p)(4 + 2p)(4 + 3p)(4 + 4p)(\omega k^4 - \gamma \delta k^3)}{p^4(\omega k - \omega)}, \tag{18.2}
\]

\[
(\omega - \beta \delta k) + 4(p^2 + 4p + 8)(\omega k^3 - \gamma \delta k^3)/p^2 = 0; \tag{18.3}
\]
and

\[ \omega^2 - s^2k^2 + (A + 2B_2\delta_1)(\alpha s^2 - \omega k)/(p + 1) = 0, \quad (19.1) \]

\[ \omega^2 - s^2k^2 - 4(\omega k^2 - \beta s^2)/p + 2a(\omega k^2 - \sigma s^2)/p + 2B_2\delta_1(\alpha s^2 - \omega k)/(p + 1) = 0. \quad (19.2) \]

\[ \omega^2 - s^2k^2 + 2B_2\delta_2(\alpha s^2 - \omega k)/(p + 1) + 16(\beta s^2 - \omega k^3)/p^2 - 4a(\sigma s^2 - \omega k^2)/p = 0. \quad (19.3) \]

It follows from Eq. (17.1) that

\[ \frac{w}{k} = \frac{Bp^2\delta_2 + 4k^2}{2p^2} + \frac{(B\delta_2/2 + 2k^2/p^2)^2 + s^2 - (B\alpha s^2 + 4\beta s^2/p^2)}{2p^2}. \quad (20) \]

From Eqs. (18.1) and (18.3), we get

\[ \omega = \frac{p^2(\beta s - w)}{2(p^2 + 4p + 8)(w - s)}, \quad (21) \]

where \( \omega \) satisfies the equation

\[ \omega^3 + \left( \frac{4(p^2 + 4p + 4)}{(p^2 + 4p + 8)^2} - \gamma s \right) \omega^2 - \left( \frac{8(p^2 + 4p + 4)\beta s}{(p^2 + 4p + 8)^2} + s^2 \right) \omega \\
+ 4\left( \frac{p^2 + 4p + 4}{(p^2 + 4p + 8)^2} + \gamma s \right) = 0. \quad (22) \]

Let

\[ (s^2\beta^2 - 1)(\beta - \gamma) < 0, \]

then there exist a real solution \( \omega^* \) to Eq. (22) such that \( \frac{\beta s - \omega^*}{\omega^* - s} > 0 \). It is easily found from Eqs. (19) with \( p = 1 \) that

\[ B_2 = \frac{\omega^2 - s^2k^2 - 4a(\sigma s^2 - \omega k^2) + 16(\beta s^2 - \omega k^3)}{\omega k - \alpha s^2}, \quad (23.1) \]

\[ \omega = \frac{(a\sigma - 10\beta s)\delta k}{a - 10k}, \quad (23.2) \]

\[ A_2 = \frac{32(\beta s^2 - \omega k^2) - 8a(\sigma s^2 - \omega k)}{\alpha s^2 - \omega k} \quad (23.3) \]

It follows from Eqs. (19) with \( p \neq 1 \) that

\[ A_2^p = \frac{(p + 1)(\omega^2 - s^2k^2)}{\omega k - \alpha s^2}, \quad (24.1) \]
\[ \omega = \frac{\alpha p (\sigma s - w)}{(8 + 2p)(\sigma s - w)}, \]

where \( w \) satisfies the equation

\[ w^3 + \left( \frac{2s^2(2 + p)}{4 + p} - \beta s \right) w^3 - \left( \frac{4s^2(2 + p)s_k^2}{(4 + p)^2} + s^2 \right) w + \frac{2s^2(2 + p)s_k^2 s_k^2}{(4 + p)^2} + \beta s^2 = 0. \]

We thus obtain the solitary wave solutions to Eqs. (2), (3), and (4), respectively,

\[ u(x,t) = \frac{A_x}{(\alpha_1 + \alpha_2)^{1/p}} \text{sech}^{2/p} \left( k(\beta_1 + \beta_2)^{-1/2} x - \left( \frac{c_1 + c_2}{2} + \frac{\omega}{k} \right) t + \eta_0 \right), \]

and

\[ u(x,t) = \frac{A_x}{(\alpha_1 + \alpha_2)^{1/p}} \left( 1 + \exp \left[ k(\beta_1 + \beta_2)^{-1/2} x - \left( \frac{c_1 + c_2}{2} + \frac{\omega}{k} \right) t + \eta_0 \right] \right)^{-2/p}. \]

III. Discussion

In this section, we discuss the relationship between the solitons of the two wave modes and the isolated wave mode. The general KdV Eq. (5) with \( p \neq 1 \) admits the solitary wave solution

\[ u(x,t) = \left( \frac{(2 + 2p)(2 + p)\beta_1 \mu_1}{\alpha_1 p^2} \right)^{1/p} \text{sech}^{2/p} \left( \mu \left[ x - \left( \frac{c_1 + 4\beta_1 \mu_1}{p^2} \right) t + \eta_0 \right] \right). \]

From (20), we get

\[ \omega = \frac{2k^3}{p^2} + \left( sk - \frac{2\beta k^3}{p^2} \right) + O(k^5), \]

with accuracy \( O(k^5) \). The soliton (26) with \( k = \sqrt{\beta_1} \mu_1, p \neq 1 \) reduces to the soliton (29) in the absence of the other wave. Putting \( B = 0 \) in (26), we get the solitary wave solution to Eq. (2) with \( p = 1 \)

\[ u(x,t) = \frac{A}{\alpha_1 + \alpha_2} \text{sech}^{2} \left( k(\beta_1 + \beta_2)^{-1/2} \left[ x - \left( \frac{c_1 + c_2}{2} + \frac{\omega}{k} \right) t + \eta_0 \right] \right). \]
where

\[ k^2 = \frac{A(\alpha s - \omega)}{12(\beta s - \omega)}, \quad \delta = \frac{\omega}{k}, \quad \rho^2 = \frac{s^2 - A(\alpha s - \omega)}/3. \]

With accuracy \(O(A^7)\), we find that

\[ \omega = \pm \alpha + A(1 \mp \alpha)/6 \pm A^2(1 - \alpha^2)/72s \pm A^3\alpha(1 - \alpha^2)/432s^2 \]
\[ \pm A^4[-1/(16 \times 36^3) - 5\alpha^4/(128 \times 36)]/s^3 \]
\[ \pm A^5[-1/(16 \times 36^3) - 5\alpha^4/(128 \times 243)][-7\alpha^4/(3^5 \times 16^2)]/s^4 \]
\[ \pm A^6[-1/(16 \times 36^3) - 5\alpha^4/(128 \times 36 \times 54)] + 35\alpha^4/(16^2 \times 36 \times 81) \]
\[ -21\alpha^6/(3^5 \times 16 \times 64)]/s^5 \quad O(A^7). \tag{32} \]

So, with accuracy \(O(A^7)\), soliton (31) reduces to the soliton of the KdV equation

\[ u = \dot{A} sech^2(L_t)[x - (c_1 \pm \alpha_1, A/3)t)], \quad 12L_t^2 = \dot{A}\alpha_1/\beta_1, \tag{33} \]

in the absence of the other wave. Putting \(p = 1\) in the soliton (26), we obtain

\[ u(x,t) = \frac{12k^2(\beta s - \omega)}{(a_1 + a_2)(a_2 k - \omega)} \text{sech}^2 \left( k(\beta_1 + \beta_2)^{-1/2} \left[ x - \left( \frac{c_1 + c_2 + \omega}{2k} \right)t + \eta_0 \right] \right) \tag{34} \]

where

\[ \omega = \frac{4k^5 + Bk}{2} + \frac{sk}{2} - 4\beta k + \frac{\alpha k B}{2} + O(k^3) + O(B^2), \]

with accuracy \(O(k^3)\). It is easily found that the soliton (34) reduces to the soliton of the KdV equation

\[ u = (12\beta \mu^2/\alpha_1) \text{sech}^2(\mu(x - (c_1 \pm \alpha_1, B_0 \pm 4\beta \mu^2)t \pm \eta_0))t \pm B_0, \tag{35} \]

in the absence of the other wave. Let \(p = 1\) in Eq. (28), then

\[ u(x,t) = \frac{A_2}{a_1 + a_2} \left( 1 + \exp \left( 2k(\beta_1 + \beta_2)^{-1/2} \left[ x - \left( \frac{c_1 + c_2 + \omega}{2k} \right)t + \eta_0 \right] \right) \right)^2 \]
\[ \tag{36} \]

\[ \frac{B_2}{a_1 + a_2} \frac{B_2 + A_2}{a_1 + a_2} \frac{A_2}{a_1 + a_2} \frac{e^{2\eta}}{1 + e^{2\eta}}. \]

Solution (36) shows that a shock wave for the two wave mode the KdV-Burgers equation is a superposition of a soliton wave for the two wave mode the KdV and a shock wave for the two wave mode Burgers equation.
Summarizing we get the solitary wave solutions with two wave modes (2-4) and the relationship between solitons for the two wave modes and the isolated wave mode. Some other topics, such as the stability of solutions (26-28), even though difficult, are worth further investigation.

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References