Magnetic Susceptibility in the Aharonov-Bohm Experiment

T. M. Hong and F. R. Lee
Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 300, R.O.C.
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We study the magnetization $M$ and the magnetic susceptibility $X$ of both an electron and many ideal electrons on a ring and a more realistic hollow disk for the Aharonov-Bohm experiment. That is, how does the electron respond to the change of a nontrivial vector potential when the magnetic field is zero? Numerical results of $M(T)$ and $X(T)$ for the ring case are supported by analytic expressions. Similar analytic expressions for the disk are only possible when we assume, based on the qualitative resemblance of its numerical results with the ring case, the eigenenergies are separable into radial and angular parts. Although this approximation is only justified rigorously when the disk is narrow, numerical results provide evidence to extend its validity to even a wide disk as long as the electron number is not too large.

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I. Introduction

In the famous paper of Aharonov and Bohm [1], later verified by the experiment of Tonomura and collaborators [2], the vector potential $\vec{A}$ is established to be a physically meaningful quantity. We are interested in the magnetic response of an electron in such an experimental set-up [3-4], i.e., in the absence of magnetic field ($B = 0$) but there exists a nontrivial $\vec{A} \neq 0$. A positive result can be viewed as another evidence of the physical significance of $\vec{A}$. Analytic expressions can be easily obtained for a ring which shall be discussed first, but not so for a hollow disk. Based on the qualitative resemblance of the magnetic responses between these two cases, we shall introduce an approximation on the eigenvalues which facilitates us to proceed analytically. For simplicity, we neglect the interparticle interaction and the spin when considering more than one electron. Extension to particle number $N \gg 1$ is straightforward by use of the grand-canonical ensemble [5], while the spin degree of freedom can be easily included by summing over the different Zeeman energies in addition to the single particle states for the grand-partition function.

It is important to note the distinction between a Aharonov-Bohm setup and a superconducting ring in the aspect of flux quantization [6]. For the latter, the phase of the Cooper-pair wavefunction, being conjugate to the nonconserving pair number [7], is fixed. Consequently, the extra phase it picks up after moving around the ring once comes solely from the magnetic flux the ring encloses, and has to be an integer number of $2\pi$. While
for an ordinary electron in the Aharonov-Bohm setup the phase of its wavefunction is not fixed. In order to obtain a unique value for the wavefunction after moving around the ring once, we simply require the \( m \) appearing in its \( \theta \)-angle dependence \( e^{im\theta} \) to be an integer. Therefore, there is no contradiction in the arbitrariness of the strength of magnetic field we impose.

Throughout this paper, the magnetic field is chosen to lie in the \( z \)-direction and confined within the (inner) radius \( R \) of either a ring or a hollow disk on the \( z-y \) plane. The vector potential can be written as

\[
\vec{A} = \begin{cases} \frac{1}{2} B\hat{\theta}, & r < R; \\ \frac{R^2}{2\pi} B\hat{\theta}, & r \geq R. \end{cases}
\]  

In cylindrical coordinates.

II. One-dimensional case

For an electron confined to move on the ring, the Hamiltonian is

\[
H = \frac{1}{2m_e} \left( \hat{\mathbf{p}} + \frac{e}{c} \vec{A} \right)^2 = \Delta \left( \frac{1}{i} \frac{d}{d\theta} + \frac{\phi}{\phi_0} \right)^2
\]

where

\[
\Delta = \frac{\hbar^2}{2m_e R^2}, \quad \phi = \pi R^2 B; \quad \phi_0 = \frac{\hbar c}{e} = 4 \times 10^{-7} \text{ Gauss cm}^2
\]

and \( m_e \) and \( e \) are the mass and electric charge of an electron. Notice that the presence of magnetic field merely shifts the angular momentum by \( \frac{\Delta}{\phi} \), and the eigenfunctions and eigenvalues can be readily written down as

\[
\psi(\theta) = \sqrt{\frac{1}{2\pi}} e^{im\theta}; \quad E_m = \Delta \left( m + \frac{\phi}{\phi_0} \right)^2.
\]

\( E_m \) versus \( \phi/\phi_0 \) is plotted in Fig. 1, which shows the splitting of the originally degenerate states of integers \( m \) and \( -m \), and \( m \neq 0 \) states can be reproduced by shifting the \( E_0 \) state horizontally by \( m \) units. When \( \phi/\phi_0 \) equals an integer, the spectrum becomes identical to that without the magnetic field. Furthermore, the spectrum is symmetric under reflection at \( \phi/\phi_0 = \text{half integers} \).

Numerical results of the temperature dependence of the magnetic moment \( M(T) \) and the magnetic susceptibility \( X(T) \) are plotted in Figs. 2 and 3. The vanishing of both quantities at high temperatures (when \( k_B T \gg \hbar \omega \)) is expected since there is no magnetism in the classical regime [8]. Analytic expressions for \( \phi/\phi_0 \in [0, 0.5] \) are

\[
M \approx -4 \frac{\pi R^2}{\phi_0} \frac{\phi}{\phi_0} e^{-\frac{\phi}{\phi_0} \sin(2\pi \phi/\phi_0)},
\]

\[
X \approx -8 \frac{\pi R^2}{\phi_0} \frac{\phi}{\phi_0} e^{-\frac{\phi}{\phi_0} \cos(2\pi \phi/\phi_0)}
\]

at high temperatures, and
FIG. 1. Field dependence of the spectrum of an electron on a ring.

FIG. 2. Temperature dependence of the magnetization, $M(T)$, of an electron on a ring.

FIG. 3. Temperature dependence of the magnetic susceptibility, $\chi(T)$, of an electron on a ring.
\[ M \approx -\frac{\pi \hbar^2}{m_e \phi_0} \left[ \frac{\phi}{\phi_0} e^{-2\Delta / (1 - \frac{2\Delta}{\phi_0})} \right] \]  

\[ \chi \approx -\frac{1}{m_e} \left( \frac{\pi \hbar R}{\phi_0} \right)^2 \left[ 1 - 2\Delta \beta e^{-2\Delta / (1 - \frac{2\Delta}{\phi_0})} \right] \]  

at low temperatures. As shown in Fig. 3, Eq. (7) increases faster with \( T \) at higher magnetic field. A physical reason for this is that the gap between the ground state and the first excited state is smaller when \( \phi / \phi_0 \) approaches half integer numbers (see Fig. 1), which thus provides more accessible states to \( \chi \). It is interesting to note that Eq. (5) changes sign at \( \phi / \phi_0 = 0.25 \), and accordingly the paramagnetic response only appears when the magnetic field exceeds this critical value. Otherwise, \( \chi \) remains diamagnetic at all temperatures.

It is straightforward to generalize the above discussions to \( N \) electrons. For instance, Figs. 4 and 5 show the numerical results of \( M(T) \) for \( N = 2 \) and \( N = 3 \) respectively. Fig. 5 is similar to the single electron results (Fig. 2) in that \( M \) increases from a negative value, but it approaches zero at high temperatures in an oscillatory behaviour with decreasing amplitudes, which is not seen in Fig. 2. Figure 4 also shares the same oscillatory behaviour but, contrary to Figs. 5 and 2, its \( M \) value is largely positive. Figs. 6 and 7 shows \( \chi(T) \) for \( N = 2 \) and \( N = 3 \) respectively. Again Fig. 7 is similar to the single electron result (Fig. 3) except for the oscillatory behaviour at high temperatures. A new feature in Fig. 6 is that \( \chi \) can become paramagnetic for all \( \phi / \phi_0 \) values, but the maximum value of \( \chi \) here is appreciably smaller than that of Fig. 7.

To check our numerical results, we obtain the following consistent analytic expressions at low temperatures:

**FIG. 4.** Magnetization \( M(T) \) per particle for two electrons on a ring.

**FIG. 5.** Same plot as in Fig. 4 for three electrons.
To check our numerical results, we obtain the following consistent analytic expressions at low temperatures:

\[
M = \frac{\pi h^2}{m_e \phi_0} \chi \left\{ \begin{array}{ll}
\left(\frac{1}{2} - \frac{\phi}{\phi_0} - e^{-N\beta \Delta \phi_0}\right), & \text{if } N \text{ is even,} \\
\left(-\frac{\phi}{\phi_0} + e^{-N\beta \Delta \phi_0 \left(1-\frac{1}{2}\phi_0\right)}\right), & \text{if } N \text{ is odd};
\end{array} \right. 
\]  

(8)

and

\[
\chi = \frac{-1}{m_e} \left(\frac{\pi h R}{\phi_0}\right)^2 \chi \left\{ \begin{array}{ll}
1 - N \Delta \beta e^{-N\beta \Delta \phi_0}, & \text{if } N \text{ is even,} \\
1 - N \Delta \beta e^{-N\beta \Delta \phi_0 \left(1-\frac{1}{2}\phi_0\right)}, & \text{if } N \text{ is odd.}
\end{array} \right. 
\]  

(9)

High-temperature results are hard to be reduced to simple forms.

III. Two-dimensional case

Let's extend our discussion to two-dimensional electrons, i.e., allow the electron to move in a hollow disk with inner and outer radii \( R \) and \( L \). The magnetic field is confined within the inner hole, and its vector potential \( \mathbf{A} \) is still expressed by Eq. (1). With the radial part retained, the Hamiltonian can be solved to give

\[
\Psi(r, \theta) = \left[ c_1 J_\nu(kr) + c_2 N_\nu(kr) \right] \frac{1}{\sqrt{2\pi}} e^{im\theta}
\]  

(10)
where $\nu = m + \nu_0/k^2 = 2m_eE/h^2$, $m$ is an integer, and $J_\nu/N_\nu$ are the Bessel/Neumman functions of a noninteger order. By imposing the boundary conditions, $\Psi = 0$ at $r = R$ and $r = L$, we obtain

$$J_\nu(kR) \cdot N_\nu(kL) - J_\nu(kL) \cdot N_\nu(kR) = 0. \tag{11}$$

The $n$th root of Eq. (11) then determines the discrete eigenenergies of the single electron as

$$E_{n,m} = \frac{\hbar^2 k^2 n^2}{2m_e} \tag{12}$$

where $n = 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots$. Asymptotic behaviour \cite{10} of the Bessel/Neumman functions at $kr \gg 1, \nu,$

$$J_\nu(kr) \approx \sqrt{\frac{2}{\pi kr}} \cos \left(kr - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2} + \frac{\nu^2 - \frac{1}{4}}{2kr}\right) \tag{13}$$

$$N_\nu(kr) \approx \sqrt{\frac{2}{\pi kr}} \sin \left(kr - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2} + \frac{\nu^2 - \frac{1}{4}}{2kr}\right) \tag{14}$$

can be used to simplify Eq. (11) to

$$\sin \left(k_{nm}(L - R) + \frac{L - R}{LR} \frac{\nu^2 - \frac{1}{4}}{2k_{nm}}\right) = 0 \tag{15}$$

and

$$k_{nm} = \frac{1}{2} \left[\frac{n\pi}{L - R} + \sqrt{\left(\frac{n\pi}{L - R}\right)^2 + \frac{2\nu^2 - \frac{1}{4}}{LR}}\right] \tag{16}$$

The spectrum as a function of magnetic field is plotted in Fig. 8. Unlike Fig. 1 for an one dimensional electron, the curves here are no longer simple parabolic. But the properties of periodicity and reflectional symmetry with respect to $\phi/\phi_0 = \text{half integers}$ still exist. In the limit of $L \rightarrow \infty$, Eq. (11) is satisfied by arbitrary $k$, i.e., the spectrum is continuous. Therefore, we expect null magnetization and magnetic susceptibility. In the other limit of $L \rightarrow R$, say $L - R \equiv \delta \ll R$, it can be shown that $E_{nm} \approx E_n + E_m$ where

$$E_n = \frac{\hbar^2}{2m_e} \left[\left(\frac{n\pi}{\delta}\right)^2 - \frac{1}{4R^2}\right] \tag{17}$$

and $E_m$ is defined in Eq. (3). The first part, $E_n$, being independent of the magnetic field, can be called the radial energy \cite{11}. While the second part, identical to the spectrum of the one-dimensional electron on the ring, is called the angular energy. Since the level difference between $E_m$ is much smaller than that between $E_n$, each $E_n$ has many finely spaced $E_m$ levels on top of it and together look like an energy band.

When we allow the separation of the eigenenergy into radial and angular parts, the partition function can be simplified by the Poisson summation formula into
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FIG. 8. Field dependence of the spectrum of an electron in a hollow disk.

\[ Z = f_n(\beta) \sqrt{\frac{\epsilon}{\beta \Delta}} \left[ 1 + 2 \sum_{n=1}^{\infty} \cos(2\pi n \phi / \phi_0) e^{-\frac{\epsilon^2}{\beta \Delta}} \right] \]  

(18)

where the function \( f_n(\beta) \) does not depend on the magnetic field and therefore will not appear in the magnetization. At high temperatures the expressions of \( M \) and \( X \) turn out to be identical to Eqs. (4) and (5) for the one-dimensional case. Even for other temperature ranges we expect qualitatively similar behaviour for \( M \) and \( X \) to the one-dimensional electron. These analyses are supported by the numerical results of Figs. 9-12.

FIG. 9. Magnetization \( M(T) \) of an electron in a hollow disk with outer and inner radii of \( L = 2 \text{ cm} \) and \( R = 1 \text{ cm} \).

FIG. 10. Same plot as in Fig. 9 for the magnetic susceptibility \( \chi(T) \).
Figs. 13 and 14 show the $M(T)$ for two and three electrons respectively. Figs. 15 and 16 are their corresponding $X(T)$. Since these figures only involved the lowest energy band $E_{n=1}$, the qualitative similarity with the one dimensional case is expected. When $N$ is large enough to move the chemical potential into higher bands, i.e., those with $n > 1$, the situation will become more complicated.

**FIG. 11.** Emphasizing the effect of different outer radii for $M(T)$ of an electron in a hollow disk with $L = 2, 4$ cm and $R = 1$ cm.

**FIG. 12.** Same plot as in Fig. 11 for $X(T)$.

**FIG. 13.** $M(T)$ per particle for two electrons in a hollow disk with $L = 2$ cm and $R = 1$ cm.

**FIG. 14.** Same plot as in Fig. 13 for three electrons.
IV. Conclusions

We have studied the magnetic response of both an electron and many electrons on either a ring or a hollow disk for the Aharonov-Bohm experiment. Numerical results of $M(T)$ and $X(T)$ for the ring case are supported by analytic expressions. However, analytical expressions for the disk case, which turn out to be similar to the one dimensional ones, are only possible when we assumed the eigenenergy to be separable into radial and angular parts, which rigorously is only justified when the radii $L - R < R$. However, qualitative consistence with the numerical results gives evidence to extend the validity of the former analytic expressions to even $L - R \gg R$ as long as the chemical potential is smaller than the lower bound of the second band, $E_{n=2}$.

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References

[6] We thank Prof. Yong-Chuan Gao for a discussion on this point.

[9] Had $L$ been infinite, there would not be any bound state for the electron. The spectrum becomes continuous, and no magnetic response is expected (cf. Ref. 3).
