Deuteron-Deuteron Cluster Model for Studying the Ground State Energy of the $^4$He Isotope

M. R. Shojaei and N. Roshanbakht

Department of Physics, University of Shahrood, P. O. Box 36155-316, Shahrood, Iran

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The first doubly magic nucleus with the first closed shell, $1s_{1/2}$, is the $^4$He isotope, and the understanding of its structure is very important. Hence, in this paper, we study the $^4$He isotope composed of two deuteron clusters and consider the central Woods-Saxon potential between them. We selected the Dirac equation and solved it for this model using the Nikiforov-Uvarov method. At the end, the ground state energy and the wave function of the isotope $^4$He have been determined, and the results are compared with the experimental data. We obtained good values, and this proves that our model has a good prediction for the ground state.

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I. INTRODUCTION

It is interesting to consider that the nucleus is composed of the clusters. The presence of nuclear matter will affect the cluster through the Pauli exclusion effect and the potential field acting on it [1]. Bethe in 1936 predicted that nuclei are made of alpha particles, and also gave a geometrical arrangement of alpha particles inside nuclei [2]. In 1937, Wheeler [3] extended this work, and similar models were suggested concurrently by Wefelmeier [4], Weizsacker [5], and Fano [6]. Kuriyama, in 1968, supposed the deuteron cluster model in nuclei and studied the transition between the deuteron-like super state and the deuteron-like boson gas state [7]. Also, Bando, in 1981, studied how deuteron clusters behave during the scattering process and the results showed that the deuteron cluster model is reliable [1]. Recently, Armstrong considered the alpha-deuteron cluster model for $^6$Li and calculated the excitation energy of the $2^+$ and $3^+$ excited states, charge radius, and dipole moment with both the Minnesota and Volkov potentials [8–10]. The results show a very dominant alpha-deuteron structure [11].

The $^4$He isotope is the first doubly magic nucleus with the first closed shell, $1S_{1/2}$; it is exceptionally stable, and its binding energy per nucleon is 7.07 MeV/nucleon [12]. It is also very compact (a charge radius of 1.673 fm [13]), and has zero for all quantum numbers (spin and isospin), so that the understanding of its structure is very important.

In this paper, we study the $^4$He isotope composed of two deuteron clusters and consider the Woods-Saxon potential between them. The deuteron is formed by just one neutron.

*Electronic address: nroshanbakht@yahoo.com
and one proton, and is only found with $s^z = 1$. The interactions between them are commonly described by using a potential that consists of the coulomb and the nuclear potentials. These potentials are usually taken to be of the Woods-Saxon form. The spherical Woods-Saxon potential and its various modified shapes was successful in describing the nuclear mean field model [14]. Hence, we use the standard Woods-Saxon potential that is given by [15]

$$V(r) = -\frac{V_0}{1 + \exp\left(\frac{r - R_0}{a}\right)},$$  \hspace{0.5cm} (1)

where $R_0$ is the radius of a nucleus and the parameter $a$ is the thickness of the superficial layer inside. In the Dirac equation, the potentials $S(r)$ and $V(r)$ are respectively called the vector and the scalar potentials [16–18]. The reason is that the so-called scalar potential is bracketed with the mass, and the so-called vector potential goes with the energy. The solution of the Dirac equation (DE) similar to the Schrödinger equation (SE) is a very important issue for solving many problems of nuclear and high-energy physics, especially relativistic problems, and to obtain an accurate solution of them is only possible for a few potentials. Hence, we put the Woods-Saxon potential instead of the $S(r)$ and $V(r)$ in the Dirac equation, and solve it by using the Pekeris approximation [19]. The standard analytical method to solve the Schrödinger equation, Dirac equation, and Klein-Gordon equation with a variable coefficient is to expand the solution in a power series of the independent variable $r$ and then find the recursion relationships for all the expansion coefficients [20]. This method has more details to reach the solution. Numerical and analytical methods complement each other to find an exact or approximate solution of the quantum, and each would be much poorer without the other. The ab-initio method is one of the numerical methods that have been used to explore the structure of light nuclei. Recently, Pieper and Wiringa used ab-initio calculations based on the Green’s function Monte Carlo (GFMC) method for the $^4\text{He}$ ground state energy and radius obtained from the nucleon-nucleon (NN) Argonne $V_8$ potential [21, 22]. But in our article we use simple “hand-power methods”, namely analytical methods, because it more revealing to see the solution stages of the problem, and so it would be more meaningful than the numerical solution.

This paper is organized as follow. In Section II, we review the parametric generalization of the Nikiforov-Uvarov (NU) method briefly. In Section III, we introduced the Dirac equation with scalar and vector potentials with arbitrary spin-orbit coupling number $\kappa$ under the spin symmetry limit. The solution of the Dirac equation with the Woods-Saxon potential and the calculation of the ground state energy and the wave function are given in Section IV. At the end, our conclusion is given in Section V.

II. THE PARAMETRIC GENERALIZATION OF THE NIKIFOROV-UVAROV (NU) METHOD

The NU method is a powerful mathematical tool for solving the Schrödinger, Dirac, and Klein-Gordon wave equations for certain kinds of potentials. This method is based on the solutions of a general second order linear differential equation with special orthogonal
functions that starts by considering the following differential equation [23, 24]:

$$
\psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \psi_n'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi_n(s) = 0
$$

(2)

where \(\sigma(s)\) and \(\tilde{\sigma}(s)\) are polynomials, at most of second-degree, and \(\tilde{\tau}(s)\) is a first-degree polynomial. The following equation is a general form of the Schrödinger-like equation written for any potential [24]:

$$
\left[ \frac{d^2}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1 - \alpha_3 s)} \frac{d}{ds} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{[s(1 - \alpha_3 s)]^2} \right] \psi_n(s) = 0.
$$

(3)

According to the Nikiforov-Uvarov method, the energy spectrum function and the eigenfunctions become the following equations, respectively:

$$
\alpha_{2n} - (2n + 1)\alpha_5 + (2n + 1)(\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}) + n(n - 1)\alpha_3 + \alpha_7 + 2\alpha_3 \alpha_8 + 2\sqrt{\alpha_8 \alpha_9} = 0
$$

(4)

and

$$
\psi(s) = s^{\alpha_{12}} (1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} P_n^{\alpha_{10} - 1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10} - 1} (1 - 2\alpha_3 s).
$$

(5)

Here, the alpha functions are given by

$$
\alpha_4 = \frac{1}{2}(1 - \alpha_1), \quad \alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3), \quad \alpha_6 = (\alpha_5^2 + \xi_1),
$$

(6)

$$
\alpha_7 = 2\alpha_4 \alpha_5 - \xi_2, \quad \alpha_8 = \alpha_7^2 + \xi_3, \quad \alpha_9 = \alpha_3 \alpha_7 + \alpha_3^2 \alpha_8 + \alpha_6,
$$

(7)

and

$$
\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \quad \alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}),
$$

(8)

$$
\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \quad \alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}).
$$

(9)

In some problems \(\alpha_3 = 0\). For this type of problem when

$$
\lim_{\alpha_3 \to 0} (1 - \alpha_3 s) P_n^{\alpha_{10} - 1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10} - 1} = L_n^{\alpha_{10} - 1}(\alpha_{11} s)
$$

(10)

and

$$
\lim_{\alpha_3 \to 0} (1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} = e^{\alpha_{13} s}
$$

(11)

the solution given in Equation (4) becomes [23, 24]

$$
\psi_n(s) = s^{\alpha_{12}} e^{\alpha_{13} s} L_n^{\alpha_{10} - 1}(\alpha_{11} s).
$$

(12)
III. THE DIRAC EQUATION WITH THE CENTRAL POTENTIAL

The Dirac equation in the relativistic description for a single-nucleon with the mass $M$ moving in an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ can be written as [25]

\[
\begin{bmatrix}
\hat{\alpha} \cdot \hat{p} + \hat{\beta}(Mc^2 + S(r))
\end{bmatrix} \psi_{n_r,\kappa}(r) = [E - V(r)] \psi_{n_r,\kappa}(r),
\]

where $\hat{p}$ is the momentum operator, $E$ is the relativistic energy, and the coefficients $\hat{\alpha}$ and $\hat{\beta}$ are the $4 \times 4$ Dirac matrices as follows:

\[
\hat{\alpha} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\]

In order, $\sigma_i$ are the Pauli matrices and $I$ is the unit matrix. In a central potential, the Dirac Hamiltonian commutes with the total angular momentum $\hat{J}$ and $\hat{K} = -\hat{\beta}(\hat{\alpha} \cdot \hat{L} + 1)$, where $\hat{L}$ is the orbital angular momentum. The eigenvalue of the total angular momentum $\hat{J}$ is $j$, and the eigenvalues of $\hat{K}$ are $\kappa = \pm (j + 1/2)$. The positive sign is for unaligned spin, and the negative sign is for aligned spin. Therefore we can write the wave function for the four spinor in a central field:

\[
\psi_{n_r,\kappa}(r, \theta, \phi) = \frac{1}{r} F_{n_r,\kappa}(r) Y_l^m(\theta, \phi) + i G_{n_r,\kappa}(r) Y_{\tilde{l}}^m(\theta, \phi),
\]

where $F_{n_r,\kappa}(r)$ are the upper components, $G_{n_r,\kappa}(r)$ are the lower components of the Dirac spinors, and $n_r$ is the radial quantum number. Also, $Y_l^m(\theta, \phi)$ and $Y_{\tilde{l}}^m(\theta, \phi)$ are the spherical harmonic functions, and $m$ is the projection of the angular momentum on the $z$-axis. $l$ and $\tilde{l}$ are the orbital angular momentum quantum numbers representing the spin and pseudospin quantum numbers. By substituting Equation (15) into Equation (13) with the usual Dirac matrices and rearranging it, two coupled differential equations for the radial wave functions are obtained:

\[
\left( \frac{d}{dr} + \frac{\kappa}{r} \right) F_{n_r,\kappa}(r) = \frac{1}{\hbar c} \left[ Mc^2 + E_{n_r,\kappa} - \Delta(r) \right] G_{n_r,\kappa}(r),
\]

\[
\left( \frac{d}{dr} - \frac{\kappa}{r} \right) G_{n_r,\kappa}(r) = \frac{1}{\hbar c} \left[ Mc^2 - E_{n_r,\kappa} + \Sigma(r) \right] F_{n_r,\kappa}(r),
\]

where

\[
\Delta(r) = V(r) - S(r),
\]

\[
\Sigma(r) = V(r) + S(r).
\]
By eliminating $F_{n_r,\kappa}(r)$ and $G_{n_r,\kappa}(r)$ from Equation (16a) and Equation (16b), we obtain the following two second-order Schrödinger-like differential equations for the upper and lower components, respectively:

\[
\begin{align*}
&\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} - \frac{1}{\hbar^2 c^2} (M^2 + E_{n_r,\kappa} - \Delta(r))(M^2 - E_{n_r,\kappa} + \Sigma(r)) \right] G_{n_r,\kappa}(r) \\
&\quad - \left[ \frac{1}{\hbar^2 c^2} \frac{(d\Sigma(r)/dr)(d/dr - \kappa/r)}{M^2 - E_{n_r,\kappa} + \Sigma(r)} \right] G_{n_r,\kappa} = 0, \quad (18a) \\
&\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - \frac{1}{\hbar^2 c^2} (M^2 + E_{n_r,\kappa} - \Delta(r))(M^2 - E_{n_r,\kappa} + \Sigma(r)) \right] F_{n_r,\kappa}(r) \\
&\quad + \left[ \frac{1}{\hbar^2 c^2} \frac{(d\Delta(r)/dr)(d/dr + \kappa/r)}{M^2 + E_{n_r,\kappa} - \Sigma(r)} \right] F_{n_r,\kappa} = 0, \quad (18b)
\end{align*}
\]

where $\kappa(\kappa - 1) = \tilde{l}(\tilde{l} + 1)$ and $\kappa(\kappa + 1) = l(l + 1)$. We consider that the wave functions satisfy the boundary conditions $F_{n_r,\kappa}(0) = G_{n_r,\kappa}(0) = 0$ and $F_{n_r,\kappa}(\infty) = G_{n_r,\kappa}(\infty) = 0$.

**IV. THE BOUND STATE SOLUTION OF THE DIRAC EQUATION FOR THE \(^4\)HE ISOTOPE**

Under the condition of the spin symmetry limit, i.e., $\Delta(r) = C_s = \text{constant}$ [26, 27], Equation (18b) reduces to

\[
\begin{align*}
&\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - \frac{1}{\hbar^2 c^2} (M^2 + E_{n_r,\kappa} - C_s)(M^2 - E_{n_r,\kappa} + \Sigma(r)) \right] F_{n_r,\kappa}(r) = 0, \quad (19)
\end{align*}
\]

where $\kappa = l$ for $\kappa < 0$ and $\kappa = -l - 1$ for $\kappa > 0$. We consider $V(r) = S(r)$ and insert the Woods-Saxon potential in Equation (19) instead of $\Sigma(r)$:

\[
\begin{align*}
&\left[ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - \frac{1}{\hbar^2 c^2} (M^2 + E_{n_r,\kappa})(M^2 - E_{n_r,\kappa} + \frac{-2V_0}{1 + \exp \left( \frac{r - R_0}{a} \right)}) \right] F_{n_r,\kappa}(r) = 0. \quad (20)
\end{align*}
\]

From Equation (20), it is seen that the equation is a combination of the exponential and inverse square potentials for the spin-orbit coupling term, that is $\kappa(\kappa + 1)/r^2$ which cannot be solved analytically using the standard methods as SUSY or NU except for $\kappa = 0, -1$. Therefore we shall use the Pekeris approximation to the spin-orbit coupling term and solve the equation analytically [28–32]. This approximation is based on the expansion of the spin-orbit coupling term in a series of exponentials depending on the inter-nuclear distance, up to the second order terms the effective $\kappa$-dependent potential preserves the original form. If we introduce the notations as follows:

\[
r = R_0(x + 1), \quad \alpha = \frac{R_0}{a},
\]
the expansion of the spin-orbit coupling term in the Taylor series around \( x = 0 \) \((r = R_0)\) is

\[
\frac{\kappa(\kappa + 1)}{r^2} = \frac{\kappa(\kappa + 1)}{R_0^2} \frac{1}{(x + 1)^2} = \frac{\kappa(\kappa + 1)}{R_0^2} (1 - 2x + 3x^2 - 4x^3 + \cdots). \tag{22}
\]

In addition, according to the Pekeris approximation, we shall express it in the following way:

\[
\frac{\kappa(\kappa + 1)}{r^2} = \frac{\kappa(\kappa + 1)}{R_0^2} \left( C_0 + \frac{C_1}{1 + \exp(\alpha r)} + \frac{C_2}{(1 + \exp(\alpha r))^2} \right). \tag{23}
\]

It is essential that this approximation is valid only for low vibrational energy states. Also, the parameters \( C_0, C_1, \) and \( C_2 \) are constant [31, 33–36]. We expand this potential in a Taylor series around the \( x = 0 \) \((r = R_0)\) point:

\[
\frac{\kappa(\kappa + 1)}{r^2} = \delta \left( C_0 + \frac{C_1}{2} + \frac{C_2}{4} (C_1 + C_2) x + \frac{\alpha^2}{16} C_2 x^2 + \frac{\alpha^3}{48} (C_1 + C_2) x^3 - \frac{\alpha^4}{96} C_2 x^4 + \cdots \right), \tag{24}
\]

where

\[
\delta = \frac{\kappa(\kappa + 1)}{R_0^2}. \tag{25}
\]

By comparing the equal powers of \( x \) in Equation (24) with Equation (22), we can obtain the expansion coefficients \( C_0, C_1, \) and \( C_2 \) as follows:

\[
C_0 = 1 - \frac{4}{\alpha} + \frac{12}{\alpha^2}, \quad C_1 = \frac{8}{\alpha} - \frac{48}{\alpha^2}, \quad C_2 = \frac{48}{\alpha^2}. \tag{26}
\]

Now we insert Equation (23) into Equation (20) and obtain

\[
\left[ \frac{d^2}{dx^2} - \delta C_0 - \frac{\delta C_1}{1 + \exp(\alpha x)} - \frac{\delta C_2}{(1 + \exp(\alpha x))^2} \right] F_{n, \kappa}(r) - \frac{1}{\hbar^2 c^2} (M c^2 + E_{n, \kappa}) \left[ M c^2 - E_{n, \kappa} + \frac{-2V_0}{1 + \exp \left( \frac{r - R_0}{\alpha} \right)} \right] F_{n, \kappa}(r) = 0. \tag{27}
\]

If we introduce \( z = e^{\alpha x} \), we will find the following second order differential equation for the upper component of the Dirac spinor:

\[
\left\{ \frac{d^2}{dz^2} + \frac{(1 + z)}{z(1 + z)} \frac{d}{dz} - \frac{1}{z^2 \alpha^2} \left[ \delta C_0 + \frac{\delta C_1}{1 + z} + \frac{\delta C_2}{(1 + z)^2} \right] \right\} F(z) - \frac{1}{z^2 \alpha^2} \left[ \frac{1}{\hbar^2 c^2} (M c^2 + E_{n, \kappa}) (M c^2 - E_{n, \kappa} - \frac{2V_0}{1 + z}) \right] F(z) = 0. \tag{28}
\]

Comparing the previous equation with Equation (3), one can find the following parameters

\[
\alpha_1 = 1, \quad \alpha_2 = -1, \quad \alpha_3 = -1, \tag{29}
\]
\[ \xi_1 = \frac{\delta C_0}{\alpha^2}, \tag{30} \]

\[ \xi_2 = -\frac{2\delta C_0 + \delta C_1 - 2V_0(Mc^2 + E_{n,\kappa})}{\hbar^2 c^2 \alpha^2}, \tag{31} \]

\[ \xi_3 = \frac{\delta C_0 + \delta C_1 + \delta C_2 + (Mc^2 + E_{n,\kappa})(Mc^2 + E_{n,\kappa} - 2V_0)}{\hbar^2 c^2 \alpha^2}. \tag{32} \]

Now, we use Equation (4) in the NU method to find the energy eigenvalue functions:

\[ -n - \frac{1}{2}(2n + 1) + (2n + 1) \left( \sqrt{\xi_1 + \xi_2 + \xi_3 + \frac{1}{4}} - \sqrt{\xi_3} \right) \]

\[ -n(n - 1) - \xi_2 - 2\xi_3 + 2\sqrt{\xi_3} \left( \xi_1 + \xi_2 + \xi_3 + \frac{1}{4} \right) = 0. \tag{33} \]

Thus the solution of this algebraic equation with respect to \( E_{n,\kappa} \) can be obtained in terms of particular values of \( n \) and \( \kappa \). To calculate the ground state energy of the \(^4\text{He}\) isotope, \( 0^+ \), we must select \( n = 1 \) and \( \kappa = \lambda = 0 \). The approximate ground state energy eigenvalues (in units of MeV) of the standard Woods-Saxon potential, between two deuterons within the spin symmetry limit for several values of \( V_0, a, \) and \( R_0 \) were calculated, and the results are shown in Table I. The measured value of the binding energy in \(^4\text{He}\) is equal to \(-28.295\) MeV [37] and with the second set of parameters we obtain for it \(-28.421\) MeV. It’s a good value, so the ground state band of \(^4\text{He}\) is well described by our method. By using Equation (5), the eigen-functions of \(^4\text{He}\) are found:

\[ F_{n,\kappa}(z) = z^{\sqrt{\xi_3}}(1 + z)^{\frac{1}{2}} \sqrt{\frac{1}{4} + \xi_1 + \xi_2 + \xi_3} \left( 2^{\sqrt{\xi_3} - 2} \sqrt{\frac{1}{4} + \xi_1 + \xi_2 + \xi_3} \right) P_n \left( \frac{1}{4} + \xi_1 + \xi_2 + \xi_3 \right)(1 + 2z), \tag{34} \]

of particular values of \( n \) and \( \kappa \).

<table>
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<th>( V_0 ) (MeV)</th>
<th>( R_0 ) (fm)</th>
<th>( a ) (fm)</th>
<th>( E_{\text{Cal.}} ) (MeV)</th>
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</tr>
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</table>
or equivalently:
\[
F_{n,\kappa}(r) = e^{\sqrt{3}(r-R_0)/a}(1 + e^{(r-R_0)/a}) \frac{1}{2} \left[ \sqrt{\frac{1}{4} + \xi_1 + \xi_2 + \xi_3} \right] \\
\times \left[ 2\sqrt{\xi_1} - 2\sqrt{\frac{1}{4} + \xi_1 + \xi_2 + \xi_3} \right] \left( 1 + 2e^{(r-R_0)/a} \right). \tag{35}
\]

Finally, the lower component of the Dirac spinor can be calculated as
\[
G_{n,\kappa}(r) = \frac{\hbar c}{Mc^2 + E_{n,\kappa}} \left( \frac{d}{dr} + \frac{\kappa}{r} \right) F_{n,\kappa}(r). \tag{36}
\]

In Figure 1 the wave function of the ground state for the upper and the lower component of the Dirac spinor are plotted. We can observe that the boundary conditions are satisfied completely.

V. CONCLUSIONS

In the present paper we have studied the $^4$He isotope in the two deuteron cluster model. The interactions between two deuterons are commonly described by using a potential that consists of the Coulomb and the nuclear potentials, so we selected the Woods-Saxon potential between them. Then, the Dirac equation has been solved by the NU method. We calculated the ground state energy and wave function of the $^4$He isotope and compared it with the experimental data. We obviously have good values of the energy obtained for the ground state, and the wave functions satisfy the boundary conditions too.

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