By analyzing the potential energy surfaces (PESs) within the interacting boson model (IBM), the first and second order shape phase transitions (SPTs) have been investigated for even-even heaviest nuclei in the A\(\sim\)100 mass region with neutron number N \(\geq\) 52. I considered the consistent-Q Hamiltonian with control parameter within the IBM coherent state formalism. The SPTs are explored with variation of the control parameter. The validity of the model is examined for two neutron-rich isotopic chains, Zirconium (\(^{40}\)Zr) isotopes for the SPT from a spherical vibrator U(5) to an axially deformed rotor SU(3) called the X(5) symmetry, and Ruthenium (\(^{44}\)Ru) isotopes for the SPT from a spherical vibrator U(5) to a \(\gamma\)-soft rotor O(6) called the E(5) symmetry. Relatively flat PESs are obtained for nuclei showing the E(5) symmetry, while in nuclei corresponding to the X(5) case, PESs with a bump are obtained.

I. INTRODUCTION

It is now well documented that the nuclear ground state shapes exhibit sudden changes with varying neutron numbers [1–23]. In particular, the signatures of the transitions between spherical and various deformed shapes clearly follow from spectroscopic data characterizing some chains of nuclei [24–29].

During the last decade there have been various theoretical investigations of the A\(\sim\)100 mass region. This transitional region is of particular interest because a rapid change in the structure of zirconium isotopes from mass number A = 98 onwards is observed, i.e., a rather sudden change from spherical to well deformed shapes [30]. The neutron-rich nuclei in this region have an abrupt change in shape from spherical (N = 58) to highly deformed (N = 60) nuclei, so that close neighbors have completely different structure.

It has been proved in [31] that the nucleus \(^{96}\)Zr (Z = 40, N = 56) is spherical, whereas the nearby nucleus \(^{98}\)Sr (Z = 38, N = 60) is highly deformed [32], because they have different total boson number N. In the determination of N, the valence number counting is always done relative to the nearest closed shells. For example, the nucleus \(^{96}\)Zr\(_{56}\) has 10 valence proton holes (relative to Z = 50) and 6 neutrons (relative to N = 50), and so, the boson number is N = N\(_{\pi}\) + N\(_{\nu}\) = 5 + 3 = 8. Similarly, \(^{98}\)Sr\(_{66}\) has N = 6 + 5 = 11 and \(^{110}\)Ru\(_{66}\) has

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N = 3 + 8 = 11 bosons.

In Ref. [1], the author investigated the evolution of the nuclear shape transition from spherical to axially rotational shapes using the coherent state formalism of the full interacting boson model (IBM) [33] Hamiltonian in Casimir form. The validity of such a model is examined for shape transitions in the rare-earth region. The purpose of the present paper is to gain more information and insight about the nature of the shape phase transition in the A~100 mass region using a simplified three-parameter Q-consistent IBM-1 Hamiltonian. This model approximates the interacting many-fermion problem using as the major degrees of freedom N pairs of valence nucleons that are treated as bosons, carrying angular momentum either 0 (the s-bosons) or 2 (the d-bosons). The model is very appropriate in order to describe even-even medium-mass, heavy nuclei, and transitional nuclei. The IBM contains a vibrational U(5) limit and rotational SU(3) limit as well as the O(6) symmetry describing γ-unstable shapes. The original U(5), SU(3), and O(6) dynamical symmetries of the IBM have played a pivotal role in nuclear structure, given their simplicity and strong appeal arising from their group theoretical properties. The shape transitions correspond to the breaking of these dynamical symmetries with changing the number of nucleons.

Critical point symmetries [17, 34] corresponding to shape phase transitions in nuclear structure are recently receiving considerable attention. The X(5) [34] and E(5) [17] critical point symmetries correspond to special solutions of the geometric Hamiltonian of Bohr and Mottelson [35], in both of which an infinite square well potential in the quadrupole degree of freedom is assumed. In X(5), corresponding to the transition from vibrational U(5) to axially symmetric prolate SU(3) nuclei, a potential function of the intrinsic shape variable β and γ is used, while in E(5), related to the transition from vibrational U(5) to γ-unstable O(6) nuclei, a γ-independent is assumed.

In Sections II, III, IV, I recall the well known basic formalism for the potential energy surfaces which are given by calculating the value of the Hamiltonian in the coherent intrinsic state. An analysis of the function of 2-parameters β, γ was used to investigate if the shape is spherical (has one end) or is deformed (has more than one end). Calculating the control parameter (η) helps in deducing the critical points and to discuss the transition of some nuclei in the A~100 mass region from U(5) to SU(3) and from U(5) to O(6).

The paper is organized as follows: In Section II, the extended Q-consistent sd-IBM1 Hamiltonian is constructed and diagonalized by using the intrinsic coherent state to get the total energy surface; the critical points are discussed in the three dynamical symmetries U(5), SU(3), and O(6) of the IBM. The PESs and the location of the critical points in the shape transitions U(5)–SU(3) and U(5)–O(6) are identified in Section III and Section IV, respectively. In Section V, the numerical results are presented for realistic Zr and Ru isotopic chains which evolve from spherical to well deformed and from spherical to γ-unstable respectively when moving from lighter to the heavier isotopes. In the final Section VI the conclusion is presented.
II. SHAPE PHASE TRANSITIONS IN THE INTERACTING BOSON MODEL

In the framework of the interacting boson model (IBM) \[33\], the nuclear structure of an even-even nuclei is described within the U(6) symmetry, possessing the U(5), SU(3), and O(6) limiting dynamical symmetries, suitable for vibrating nuclei, axially symmetric deformed rotating nuclei, and γ-unstable rotating nuclei, respectively. Now, we want to study how the potential energy surface (PES) evolves when we cross the IBM through its symmetry limits, i.e., the first shape phase transition (SPT) between spherical and prolate nuclei, the so called X(5) symmetry \[34\] as well as the second order SPT from spherical to γ-soft nuclei identified as the E(5) symmetry \[17\]. This correspondence is readily established if we write the extended Q-consistent sd IBM-1 \[36, 37\] Hamiltonian in the form:

$$H_B(N, \eta, \chi, K) = C \left[ \hat{n}_d - \frac{1 - \eta}{N} \hat{Q}(\chi) \cdot \hat{Q}(\chi) \right] + KL \cdot \hat{L},$$  \hspace{1cm} (1)

where \(C\) is a scaling factor and \(N\) is the number of valence bosons. Here \(\hat{n}_d\), \(\hat{L}\), and \(\hat{Q}(\chi)\) are the d-boson number operator, the angular momentum operator, and the quadrupole operator, respectively:

$$\hat{n}_d = d^\dagger \cdot \tilde{d},$$  \hspace{1cm} (2)

$$\hat{Q}(\chi) = [S^\dagger \times \tilde{a} + d^\dagger \times \tilde{S}] + \chi[d^\dagger \times \tilde{d}]^{(2)},$$  \hspace{1cm} (3)

$$\hat{L} = \sqrt{10}[d^\dagger \times \tilde{d}]^{(1)},$$  \hspace{1cm} (4)

with \([d^\dagger \times \tilde{d}]\) standing for the \(l\) tensor coupling of the d-boson creation and annihilation operators \(\tilde{d}_\mu = (-1)^\mu d_{-\mu}\), where \(\mu = -l, -l+1, \ldots, +l\) is the angular momentum projection and (dot) denoting the scalar product.

For \(K = 0\), the above Hamiltonian contains two parameters, the control parameter \(\eta\) and the structure parameter of the quadrupole operator \(\chi\), with the parameter \(\eta\) ranging from 0 to 1, and the parameter \(\chi\) ranging from 0 to \(-\sqrt{7}/2\). In this parametrization, the U(5) limit corresponds to \(\eta = 1\), the O(6) limit to \(\eta = 0, \chi = 0\), and the SU(3) limit to \(\eta = 0, \chi = -\sqrt{7}/2\).

In the intrinsic frame formalism, the ground state of a nucleus with \(N\) bosons can be expressed as a boson condensate with specific quadrupole deformation parameters \(\beta\) and \(\gamma\) in the form \[33\]:

$$|g, N, \beta, \gamma\rangle = \frac{1}{\sqrt{N!}} \left( \frac{1}{\sqrt{1 + \beta^2}} \left[ S^\dagger + \beta \cos \gamma d_0^\dagger + \frac{1}{\sqrt{2}} \beta \sin \gamma (d_2^\dagger + d_{-2}^\dagger) \right] \right)^N |0\rangle,$$  \hspace{1cm} (5)

where \(|0\rangle\) denotes the boson vacuum. Here \(\beta \geq 0\) and \(0 \leq \gamma \leq \pi/3\) are the intrinsic shape parameters.
The corresponding PESs as a function of the intrinsic shape variables \( \beta \) and \( \gamma \) is given by

\[
E(N, \eta, \chi, K, \beta, \gamma) = C \left[ \eta \frac{N}{1+\beta^2} \beta^2 - \frac{(1-\eta)}{N} \left[ \frac{N}{1+\beta^2} \left[ 5 + (1+\chi^2)\beta^2 \right] + \frac{N(N-1)}{(1+\beta^2)^2} \right] \right. \\
\times \left[ 4\beta^2 + \frac{2}{7} \chi^2 \beta^4 - 4 \sqrt{\frac{2}{7}} \chi \beta^3 \cos 3\gamma \right] \bigg] + \frac{6NK}{1+\beta^2} \beta^2.
\]  

Equation (6) can be written in another form as

\[
E(N, \eta, \chi, K, \beta, \gamma) = \frac{1}{(1+\beta^2)^2} \left[ A_2 \beta^2 + A_3 \beta^3 \cos 3\gamma + A_4 \beta^4 \right] + a_0 + 6NK \frac{\beta^2}{1+\beta^2},
\]  

with

\[
A_2 = \eta N - (1-\eta)(4N + \chi^2 - 8),
\]

\[
A_3 = (1-\eta)(N-1)4\sqrt{\frac{2}{7}} \chi,
\]

\[
A_4 = \eta N - (1-\eta) \left[ \frac{2N+5}{7} \chi^2 - 4 \right],
\]

\[
a_0 = -5(1-\eta).
\]

For \( K = 0 \), the critical point in the above expression, Equation (7), is given by the value of \( \eta \) where the coefficient of \( \beta^2 \) vanishes, i.e., \( A_2 = 0 \), yielding

\[
\eta_{\text{critical}}(\chi) = \frac{4N - (8 - \chi^2)}{5N - (8 - \chi^2)}. \quad (8)
\]

At this value, the second derivative of Equation (7) with respect to \( \beta \) at \( \beta = 0 \) changes its sign, which means that \( \beta = 0 \) becomes a local minimum. The critical point Equation (8) depends on \( \chi \), it changes between

\[
\eta_{\text{critical}} \left( -\frac{\sqrt{7}}{2} \right) = \frac{16N - 25}{20N - 25} \quad \text{for the U(5) – SU(3) transition}
\]

and

\[
\eta_{\text{critical}}(0) = \frac{4N - 8}{5N - 8} \quad \text{for the U(5) – O(6) transition}
\]

which for the large-\( N \) limit give \( \eta = 4/5 \).
II-1. The vibrational limit
This limit is based on the chain of subalgebras $U(5) \supset O(5) \supset O(3)$. The simplest case is the linear Casimir operator (number operator) of $U(5)$:

$$\hat{H}(U(5)) = \varepsilon \hat{n}_d.$$  \hspace{1cm} (9)

The ground state is a condensate of $s$ bosons. The PES is $\gamma$-independent and has the form

$$E(\beta) = \varepsilon \frac{N}{1 + \beta^2} \beta^2.$$  \hspace{1cm} (10)

Despite the peculiar form of the Hamiltonian, the function $E(\beta)$ provides the correct information about the equilibrium shape, since its minimum, which in the present case is at $\beta = 0$ ($\varepsilon \geq 0$).

II-2. The rotational limit
This limit is based on the group chain $U(6) \supset SU(3) \supset O(3)$ with the Hamiltonian

$$\hat{H}(SU(3)) = K \hat{L} \cdot \hat{L} + a_2 \hat{Q} \left( -\frac{\sqrt{7}}{2} \right) \cdot \hat{Q} \left( -\frac{\sqrt{7}}{2} \right).$$  \hspace{1cm} (11)

For the PES we obtain

$$E(\beta, \gamma) = K \frac{6N}{1 + \beta^2} \beta^2 + a_2 \left[ \frac{N}{1 + \beta^2} \left( 5 + \frac{11}{4} \beta^2 \right) + \frac{N(N-1)}{(1 + \beta^2)^2} \left( 4\beta^2 + \frac{1}{2} \beta^4 + 2\sqrt{2} \beta^3 \cos 3\gamma \right) \right].$$  \hspace{1cm} (12)

For a special case, if $K = 0$ and eliminating the contributions of the one-body terms of the quadrupole- quadrupole interaction, then

$$E(\beta, \gamma) = a_2 \frac{N(N-1)}{(1 + \beta^2)^2} \left[ 4\beta^2 + \frac{1}{2} \beta^4 + 2\sqrt{2} \beta^3 \cos 3\gamma \right].$$  \hspace{1cm} (13)

The equilibrium values are obtained by solving $\partial E/\partial \beta = \partial E/\partial \gamma = 0$ to give

$$\beta_e = \sqrt{2} \text{ and } \gamma_e = 0 \text{ for } \chi = -\sqrt{7}/2,$$

$$\beta_e = \sqrt{2} \text{ and } \gamma_e = 60 \text{ for } \chi = +\sqrt{7}/2,$$

corresponding to prolate and oblate deformed shapes, respectively.

II-3. The $\gamma$-unstable limit
This limit of the IBM is based on the group chain $O(6) \supset O(5) \supset O(3)$ with the Hamiltonian

$$\hat{H}(O(6)) = a\hat{Q}(0) \cdot \hat{Q}(0).$$  \hspace{1cm} (14)
The PES is
\[ E(\beta) = a \left[ \frac{N}{1 + \beta^2} (5 + \beta^2) + \frac{N(N - 1)}{(1 + \beta^2)^2} \frac{\beta^2}{4} \right]. \] (15)

The potential is independent of \( \gamma \), which indicates the \( \gamma \)-unstable character of the dynamics in the \( \gamma \)-direction.

If we eliminate the contribution of the one-body terms of the quadrupole-quadrupole interaction, then
\[ E(\beta) = a \frac{N(N - 1)}{(1 + \beta^2)^2} \frac{\beta^2}{4}. \] (16)

The equilibrium value is given by \( \beta_e = 1 \) corresponding to the \( \gamma \)-unstable deformed shape.

### III. SHAPE PHASE TRANSITION FROM A SPHERICAL VIBRATOR U(5) TO AN AXIALLY DEFORMED ROTOR SU(3)

In the following, we adopt here \( \chi = -\sqrt{7}/2 \) and \( \gamma = 0 \) in the transitional Hamiltonian Equation (1). The corresponding scaled PES as a function of the variational parameters \( \beta \) and \( \gamma \) is given by
\[ E(N, \eta, K, \beta, \gamma) = \frac{N \beta^2}{1 + \beta^2} \left[ \eta + \frac{9}{4} (1 - \eta) + 6K \right] - \frac{(N - 1)}{(1 + \beta^2)^2} (1 - \eta) \left[ 4\beta^2 + \frac{1}{2} \beta^4 + 2\sqrt{2}\beta^3 \right]. \] (17)

In Figure 1, we show the PESs for \( N = 10 \) for the most significant values of \( \eta \). We observe the evolution from a spherical potential \( \eta = 1 \), whose minimum is found at \( \beta = 0 \), to potentials with well-deformed minima \( \eta = 0.72 \) and \( \eta = 0.74 \). For intermediate \( \eta \) values one finds a set of potential energy curves which are practically degenerate along the prolate axis in the interval \( \beta \in [0, 0.4] \). These curves show two minima, a spherical and a prolate deformed one, and can be found only in a very reduced range of values for the control parameter \( 0.749 \leq \eta \leq 0.745 \). In particular for \( \eta = 0.7453 \), the spherical and the prolate deformed minima are degenerate, and this condition defines precisely the critical point of a first order phase transition where the order parameter is the deformation \( \beta \).

### IV. SHAPE PHASE TRANSITION FROM A SPHERICAL VIBRATOR U(5) TO A \( \gamma \)-SOFT ROTOR O(6)

The transition between the vibration and \( \gamma \)-unstable deformed shape can be studied by choosing \( \chi = 0 \). The corresponding scaled PES is given by
\[ E(N, \eta, K, \beta) = b \frac{1}{1 + \beta^2} + A_1 \frac{\beta^2}{1 + \beta^2} + A_2 \frac{\beta^2}{(1 + \beta^2)^2}, \] (18)
FIG. 1: Representation of the PESs $E(\beta)$ versus the deformation parameter $\beta$ (for $N = 10$, $\gamma = 0$, $K = 0$, $\chi = -\sqrt{7}/2$) given by the IBM to describe the U(5)–SU(3) transition at different values of the control parameter $\eta$. 

with

$$b = -5(1 - \eta),$$

$$A'_1 = \eta N - (1 - \eta) + 6 NK,$$

$$A'_2 = -4(1 - \eta)(N - 1).$$

Consequently the corresponding PESs are $\gamma$-independent.

In Figure 2, the corresponding PESs with arbitrary units are plotted for $N = 20$ as a function of the shape parameter $\beta$. Three values of the control parameter $\eta$ are presented, one of the critical value $\eta_{\text{critical}} = 0.782((4N - 8)/(5N - 8))$, one above ($\eta = 0.9$) and one below ($\eta = 0.7$) that value.
V. APPLICATION TO THE A\~{}100 MASS REGION

Firstly we will study the first order SPTs between spherical and prolate deformed nuclei U(5)-SU(3) (X(5) symmetry) by analyzing the PESs of the Zirconium isotopes $^{94-104}$Zr. In Figure 3, we present the PESs for different Zr isotopes. We adopt $\chi$ at $-4/5$ and the rest of the parameters are listed in Table I. It is clearly observed that $^{94-96}$Zr remains spherical. Note that the flat PES of $^{94}$Zr appears to be due to the low number of neutron bosons $N_\nu = 2$ and to the fact that the depth of the PES is proportional to $N(N-1)$. $^{98,100}$Zr are special cases because these nuclei appear to be situated very close to the critical region where two minima coexist, one spherical and one deformed. One notices that the PES is very flat, which is caused by the fact that a deformed minimum and a spherical maximum appear that are almost degenerate. This indicates that one is close to the critical area. Finally, $^{102,104}$Zr become well deformed.

Second, we applied the second order phase transition from spherical to $\gamma$-soft nuclei (identified as the E(5) symmetry) on the neutron-rich Ruthenium isotopic chain $^{92-114}$Ru. We adopt $\chi$ and $K$ at zero and the control parameter $\eta$ is listed in Table II. In Figure 4 the corresponding PESs are plotted for this chain of nuclei which evolve from spherical to...
FIG. 3: PESs calculated by the IBM coherent state formalism for $^{94-104}$Zr isotopes.

TABLE I: The adopted model parameters $\eta$ and $k$ for Zr isotopes ($N_T = 5$).

<table>
<thead>
<tr>
<th>Isotope</th>
<th>$^{94}$Zr</th>
<th>$^{96}$Zr</th>
<th>$^{98}$Zr</th>
<th>$^{100}$Zr</th>
<th>$^{102}$Zr</th>
<th>$^{104}$Zr</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_\nu$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.895</td>
<td>0.886</td>
<td>0.748</td>
<td>0.66</td>
<td>0.433</td>
<td>0.398</td>
</tr>
<tr>
<td>$K$</td>
<td>0.05</td>
<td>0.17</td>
<td>0.15</td>
<td>0.0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$\gamma$-unstable shapes.

The set of parameters of the model for each nucleus in each isotopic chain is adjusted by using a computer simulated search program in order to describe the gradual change in the structure as the boson number varied. The $\chi$-test has been used to perform the fitting procedure for getting the IBM Hamiltonian parameters. The $\chi$-function is defined in the standard way as

$$\chi = \sqrt{\frac{1}{N} \sum_{i=1}^{N} [X_i \text{ (data)} - X_i \text{ (IBM)}]^2},$$

where $N$ is the number of experimental data. $X_i \text{ (data)}$ are the selected set of energy levels of the positive parity excitations $2^+_1, 4^+_1, 6^+_1, 8^+_1, 2^+_2, 3^+_1, 4^+_2, 0^+_2, 2^+_3, 4^+_3$, and the two
TABLE II: The adopted control parameter $\eta$ for Ru isotopes ($N_\pi = 3$).

<table>
<thead>
<tr>
<th>$^{92}$Ru</th>
<th>$^{94}$Ru</th>
<th>$^{96}$Ru</th>
<th>$^{98}$Ru</th>
<th>$^{100}$Ru</th>
<th>$^{102}$Ru</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_\nu$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.91</td>
<td>0.887</td>
<td>0.865</td>
<td>0.844</td>
<td>0.823</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$^{104}$Ru</th>
<th>$^{106}$Ru</th>
<th>$^{108}$Ru</th>
<th>$^{110}$Ru</th>
<th>$^{112}$Ru</th>
<th>$^{114}$Ru</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_\nu$</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.782</td>
<td>0.762</td>
<td>0.742</td>
<td>0.722</td>
<td>0.701</td>
</tr>
</tbody>
</table>

FIG. 4: PESs corresponding to the transition from spherical to deformed $\gamma$-unstable in Ru isotopes.

neutron separation energies, which are quite sensitive to nuclear structure, are defined as

$$S_{2n}(Z, N) = BE(Z, N) - BE(Z, N - 2),$$

where $BE(Z, N)$ is the binding energy of nuclei with proton number $Z$ and neutron number $N$. In calculating the two neutron separation energies an additional contribution
which is linear in the number of bosons is added:

\[ S_{2n}(NB) = A + BNB + BE(N_B) - BE(N_B - 1), \]

with the coefficients A and B being taken as constant across the isotopic chain. The experimental data for the excitation and binding energies have been taken from the National Nuclear Data Center [38].

To get the optimized model parameters which give both the spectroscopic properties and the intrinsic deformation, a fitting procedure using the above χ-test to the experimental excitation energies was performed for each nucleus. These model parameters give a satisfactory description of the experimental [38] and the theoretical [39] energy ratios in the considered nuclei. For example, I choose the level structure of the vibrator nucleus \(^{100}\)Ru which exhibits the U(5) dynamical symmetry [40]; the interesting feature of this nucleus is that it is located far away from semi-closed shells. To get the characteristics of the collectivity in this nucleus, the behavior of the energy ratios \(R_{1/2} = E(I_1)/E(2_1)\) was examined. In Table III, I give my calculated IBM energy ratios for the \(I^\pi = 2^+, 4^+, 6^+, 8^+, 10^+\) levels compared to the experimental data and with those of the dynamical symmetries U(5) and O(6). The agreement between the theoretical and experimental excited levels is very good, and it is clear that the nucleus \(^{100}\)Ru is well described by the U(5) dynamical symmetry.

TABLE III: The calculated energy ratios \(R_{1/2} = E(I_1)/E(2_1)\) for \(^{100}\)Ru. The experimental and the U(5) and O(6) predictions are shown for comparison.

<table>
<thead>
<tr>
<th>(I^\pi)</th>
<th>U(5)</th>
<th>Cal.</th>
<th>Exp.</th>
<th>O(6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(_1^+)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4(_1^+)</td>
<td>2</td>
<td>2.2493</td>
<td>2.2734</td>
<td>2.5</td>
</tr>
<tr>
<td>6(_1^+)</td>
<td>3</td>
<td>3.7521</td>
<td>3.8474</td>
<td>4.5</td>
</tr>
<tr>
<td>8(_1^+)</td>
<td>4</td>
<td>5.4997</td>
<td>5.6721</td>
<td>7</td>
</tr>
<tr>
<td>10(_1^+)</td>
<td>5</td>
<td>7.3334</td>
<td>7.5688</td>
<td>10</td>
</tr>
</tbody>
</table>

To test the prediction of triaxiality (γ-softness) in nuclei in the mass region \(A\sim 100\), I chose the nucleus \(^{102}\)Zr as an example. Using the master equation (7) in the framework of our simple model Equation (1), triaxial calculations are performed. The PESs cuts at γ-values of 0°, 30°, and 60° in the two-dimensional grid (\(\beta_2, \gamma\)) are plotted in Figure 5. I notice that the calculated equilibrium deformations obtained from the triaxial calculations (triaxial shape) are approximately the same as for the axial shape [39, 40].

The authors of [41, 42] studied the possible triaxiality in the A\sim 100 mass region and the role of the γ-deformation for \(^{108–112}\)Ru by using a rigid triaxial rotor model, but unfortunately this model failed to reproduce the measured excitation energies of levels with \(I > 4h\). So that in this paper I do not examine the contribution of the γ-deformation (I put \(\gamma = 0\)) and axial calculations were carried out. I hope in the next work to perform
triaxial calculations in a two-dimensional deformation grid \((\beta_2, \gamma)\) by modifying the original Hamiltonian to include the three body interaction between the d-bosons or the cubic quadrupole operator (cubic terms) to reproduce the correct triaxial systematics [43].

VI. CONCLUSION

The paper is focused on the properties of shape phase transitions between spherical and prolate deformed U(5)–SU(3) and between the spherical and \(\gamma\)–unstable shape U(5)–O(6) in the framework of the interacting boson model (sd – IBM), using the coherent state formalism. The consistent Q Hamiltonian is used and studied in the three different limits of the IBM and shape transitions between them. The evolution from spherical to deformed shapes along the chain of Zr isotopes and the evolution from spherical to \(\gamma\)-unstable shapes along the chain of the Ru isotopes are discussed. The PESs and critical points are analyzed. It is shown that the nuclei \(^{94–98}\text{Zr}\) remain spherical, \(^{100}\text{Zr}\) is the critical nucleus, while \(^{102–104}\text{Zr}\) become well deformed.

For the Ru isotopic chain, the \(^{92–98}\text{Ru}\) are vibrator like nuclei and the deformation increases with increasing neutron number. For each nucleus of each isotopic chain the model parameters have been adjusted by using a computer simulated search program.

References

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