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Dual Integral Equations Occurring in Diffraction Theory

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A new method of solving the dual integral equations is proposed. Uses are made of Erdélyi and Sneddon's Fredholm equations. The method is more satisfactory of extending the range of values of the paremeters. Probably the most interesting and central parts are change of the representations and an attemption to solve the Fredholm equation with infinite degenerate kernel.

1. INTRODUCTION

DUAL integral equations of the form

$$\int_a^\infty u^{-\nu-\mu}(u^2-a^2)^r \Psi(u) J_\mu(xu)\,du = F(x) \qquad 0<x<1, \tag{1}$$

$$\int_0^\infty \Psi(u) J_\nu(xu)\,du = G(x) \qquad x>1, \tag{2}$$

where $F(x)$ and $G(x)$ are given functions, $\Psi(x)$ is unknown, $a\geq 0, \mu, \nu$ and r are real constants have applications to diffraction theory and also to potential problems and dynamic problems in elasticity.

Many special cases have been considered by several authors(1),(2),(3),(4),(5). The case in which $\mu>\nu>r>-1$ has been considered by Burlack(6). Now we present here a method which reduces the dual integrations to a single Fredholm integral equation(7), introduced by Erdelyi and Sneddon, from the solution of which Ψ is explicitly obtainable for no restriction upon paremeter r.

2. LEMMAS

Lemma 1 (ref. 8, p. **411)**

$$\int_0^\infty u^{\nu-\mu-1} J_\nu(bu) J_\mu(cu)\,du = 0 \qquad \text{if } 0<c<b$$

provided $\mu>\nu>-1$.

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(2) E. T. Copson, Proc. Glasgow Math. Assoc. 5, 21 (1961).
(3) M. Lowengrub, I. N. Sneddon, Proc. Glasgow Math. Assoc. 6, 14 (1963).
(4) J. Burlack, Proc. Glasgow Math. Assoc. 6, 39 (1963); *ibid* 6, 117 (1963).
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(6) J. Burlack, Proc. Edinburgh Math. Soc. **13**, 179 (1962).
(7) A. Erdelyi, I. N.. Sneddon, Canada J. Math. 14, 683 (1962).

Lemma 2 (ref. 8, p. 415)

$$\int_0^\infty J_\nu(bu)\frac{J_\mu(c\sqrt{u^2+v^2})}{(u^2+v^2)^{(1/2)\mu}}u^{\nu+1}du=0 \qquad \text{if } 0<c<b$$

provided $\mu>\nu>-1$.

Lemma 3

$$\int_0^\infty u^{2\gamma+1}J_0(x\sqrt{u^2+a^2})\,J_0(y\sqrt{u^2+a^2})du=2^\gamma a^{\gamma+1}D(x,\ y,\ a,-\nu-1),$$

where $$D(x,y,a,-\gamma-1)=D(y,x,a,-\gamma-1)\equiv\sum_{n=0}^\infty(-1)^n\frac{\Gamma(n+\gamma+1)}{n!}x^{-(n+\gamma+1)}$$

$$J_{-(n+\gamma+1)}(xa)y^nJ_n(ya).$$

Proof. Using (ref. 8, p. 177(8))

$$\int_0^\infty t^{2\gamma+1}J_0(x\sqrt{u^2+a^2})J_0(y\sqrt{u^2+a^2})du$$

$$=-\frac{1}{4\pi^2}\int_{c-\infty i}^{c+\infty i}t^{-1}\exp\left\{t-\frac{y^2a^2}{4t}\right\}dt\int_{c'-\infty i}^{c'+\infty i}\tau^{-1}\exp\left\{\tau-\frac{x^2a^2}{4\tau}\right\}d\tau$$

$$\int_0^\infty u^{2\gamma+1}\exp\left\{-\frac{1}{4}\left(\frac{y^2}{t}+\frac{x^2}{\tau}\right)u^2\right\}du$$

$$=-\frac{\Gamma(\gamma+1)4^\gamma}{2\pi^2x^{2\nu+2}}\int_{c-\infty i}^{c+\infty i}t^{-1}\exp\left\{t-\frac{y^2a^2}{4t}\right\}dt$$

$$\int_{c'-\infty i}^{c'+\infty i}\tau^\gamma\left(1+\frac{yt}{x\tau}\right)^{-(\gamma+1)}\exp\left\{\tau-\frac{x^2a^2}{4\tau}\right\}d\tau,$$

expanded $\left(1+\frac{yt}{x\tau}\right)^{-(\gamma+1)}$ in power series and using (ref. 8, p. 177 (8)) again, then the integration is

$$2^\gamma a^{\gamma+1}\sum_{n=0}^\infty(-1)^n\frac{\Gamma(n+\gamma+1)}{n!}x^{-(n+\gamma+1)}J_{-(n+\gamma+1)}(xa)\,y^nJ_n(ya).$$

3. ERDÉLYI AND SNEDDON'S EQUATIONS

Erdélyi and Sneddon's have reduced the dual integral equations (ref. 7, (1.4))

$$\int_0^\infty u^{-2\alpha}\{1+K(u)\}\Psi(u)J_\mu(xu)du=F(x) \qquad 0<x<1, \tag{3}$$

$$\int_0^\infty u^{-2\beta}\Psi(u)\,J_\mu(xu)du=G(x) \qquad x>1, \tag{4}$$

into $$h_1(u)+\int_0^1K_0(u,v)h_1(v)dv=R_0(u) \qquad 0<u<1, \tag{5}$$

where h_1 is the unknown function, K_0 and R_0 are known functions given by (ref. 7, (4.3)) and (ref. 7, (4.2)) respectively as follows:

$$K_0(u,v)=\left(\frac{u}{v}\right)^{(1/2)(\alpha+\beta)}\int_0^\infty J_{\mu+\beta-\alpha}(2\sqrt{uz})J_{\mu+\beta-\alpha}(2\sqrt{vz})K(2\sqrt{z})dz \quad (6)$$

and
$$R_0=I_{(1/2)\mu+\alpha,\ \beta-\alpha}f-S_{(1/2)\mu-\alpha,\ \beta+\alpha}\{kS_{(1/2)\mu+\beta,\ -\alpha-\beta}(K_{(1/2)\mu-\alpha,\ \alpha-\beta}g)\}, \quad (7)$$

where according to (ref. 7, (1.6)) and (ref. 7, (1.2))

$$\begin{cases} k(u)=K(2\sqrt{u}) & f(u)=2^{2\alpha}u^{-\alpha}F(\sqrt{u}) \\ g(u)=0 \text{ if } u<1 & g(u)=2^{2\beta}u^{-\beta}G(\sqrt{u}) \text{ if } u\leq 1, \end{cases} \quad (8)$$

and I, K and S are integral operators, discussed in (ref. 7, (2.2)) **to (ref.** 7, (2.10)), defined as follows

$$I_{\eta,\alpha}f(x)=\frac{x^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}y^\eta f(y)dy \quad (9)$$

$$K_{\eta,\alpha}f(x)=\frac{x^\eta}{\Gamma(\alpha)}\int_x^\infty (y-x)^{\alpha-1}y^{-\eta-\alpha}f(y)dy \quad (10)$$

$$S_{\eta,\alpha}f(x)=x^{-(1/2)\alpha}\int_0^\infty y^{-(1/2)\alpha}J_{2\eta+\alpha}(2\sqrt{xy})f(y)dy. \quad (11)$$

Recently, Love[9] has discussed the case with K(u) $=e^{\pm a\rho}$ occurring for Nicholson's problem and hydrodynamics.

4. CHANGE OF THE REPRESENTATIONS

Since the dual intregral equations (1) and (2) are linear in Ψ we may write

$$\Psi(u)=\Psi_1(u)+\Psi_2(u) \quad (12)$$

as follows

$$\int_a^\infty u^{-\nu-\mu}(u^2-a^2)^\tau\Psi_1(u)J_\mu(xu)du=F(x) \qquad 0<x<1 \quad (13)$$

$$\int_0^\infty \Psi_1(u)J_2(xu)du=0 \qquad x>1 \quad (14)$$

and
$$\int_a^\infty u^{-\nu-\mu}(u^2-a^2)^\tau\Psi_2(u)J_\mu(xu)du=0 \qquad 0<x<1 \quad (15)$$

$$\int_0^\infty \Psi_2(u)J_\nu(xu)du=G(x) \qquad x>1. \quad (16)$$

On comparing (14) with Lemma 1, we see that a particular solution of (14) is the function $u^{\nu-\nu+1}J_\mu(yu)$, with $y\leq 1, \mu>\nu>-1$, and so, by superposition, we are led to the expression

$$\Psi_1(u)=u^{\nu-\mu+1}\int_0^1 y\varphi_1(y)J_\mu(uy)dy, \quad (17)$$

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(8) G. N. Watson, *Theory of Bessel Junctions*, 2nd ed. 1958.
(9) E. R. Love, Canada J. Math. 15, 631 (1963).

substituting (17) into (14), inverting the order of intergration and using Lemma 1, we verify that (17) satisfy (14).

Now by the Hankel inversion theorem applied to (17) shows that

$$\int_0^\infty u^{-(\nu-\mu)}\Psi_1(u)J_\mu(yu)du=0 \qquad y>1$$

provided $\mu>\nu>-1$. Then (14) is equivalent to

$$\int_0^\infty u^{-(\nu-\mu)}\Psi_1(u)J_\mu(xu)du=0 \qquad x>1. \tag{18}$$

Introduce a step function

$$S(u-a)=\begin{cases}0 & \text{if } u<a\\ 1 & \text{if } u\geq a,\end{cases}$$

then (13) and (14) become

$$\int_0^\infty u^{-(\nu+\mu)}S(u-a)\,(u^2-a^2)^r\Psi_1(u)J_\mu(xu)du=F(x) \qquad 0<x<1 \tag{19}$$

$$\int_0^\infty u^{-(\nu-\mu)}\Psi_1(u)J_\mu(xu)du=0 \qquad x>1. \tag{20}$$

By the same arguments, using Lemma 2 and define

$$\Psi_2=u^{\nu+1}(u^2-a^2)^{(1/2)\lambda-r}\int_1^\infty y\varphi_2(y)J_\lambda(y\sqrt{x^2-a^2})dy,$$

we reduce (15) and (16) equivalently into

$$\int_0^\infty u^{(3/2)\nu}S(u-a)(u^2-a^2)^r\Psi_2(u)J_\nu(xu)du=0 \qquad 0<x<1 \tag{21}$$

$$\int_0^\infty \Psi_2(u)J_\nu(xu)du=G(x) \qquad x>1. \tag{22}$$

Since the subscripts for Bessel function in the pair of (19), (20) and (21), (22) are same, then Erdélyi and Senddon ' s Fredholm equation (5) may be used.

5. THE SOLUTION

In the present paper we shall consider the equations with

$$G(x)=0 \qquad x>1,$$

i.e. in eqs. (19) and (20) for $\Psi=\Psi_1$.

On comparing (19) and (20) with (3) and (4) for $G(x)=0$ we see that

$$\begin{aligned}&1+K(u)=S(u-a)(u^2-a^2)^r,\\ &\alpha=\frac{1}{2}(\nu+\mu), \qquad \beta=\frac{1}{2}(\nu-\mu).\end{aligned} \tag{23}$$

And we now simplify (6), first making the substitutions

$$x=\sqrt{u},\ y=\sqrt{v},\ t=2\sqrt{z}$$

$$K_0(x^2,\ y^2)=\frac{1}{2}\left(\frac{y}{x}\right)^\nu\int_0^\infty J_0(xt)J_0(yt)\left\{-1+S(t-a)(t^2-a^2)^\gamma\right\}tdt$$

$$=-\frac{1}{2}\left(\frac{y}{x}\right)^\nu\int_0^\infty tJ_0(xt)J_0(yt)\,dt$$

$$+\frac{1}{2}\left(\frac{y}{x}\right)^\nu\int_0^\infty t^{2\gamma+1}J_0(x\sqrt{t^2+a^2})J_0(y\sqrt{t^2+a^2})\,dt,$$

using Lemma 3, we get

$$K_0(x^2, y^2)=2^{\gamma-1}a^{\gamma+1}\left(\frac{y}{x}\right)^\nu D(x, y, a,-\gamma-1). \tag{24}$$

Instead of h_1, we take ϕ as unknown function

$$\phi(x)=x^{\nu+1}h_1(x^2). \tag{25}$$

Now (5) becomes

$$\phi(x)+\int_0^1 K(x, y)\phi(y)dy=R(x) \qquad 0<x<1, \tag{26}$$

where
$$K(x, y)=2y^{-\nu}x^{\nu+1}K_0(x^2, y^2)$$
$$=2^\gamma a^{\gamma+1}xD(x, y, a,-\gamma+1), \tag{27}$$

$$R(x)=x^{\nu+1}R_0(x^2). \tag{23}$$

Next we simplify (7) under the condition $G(x)=0$. Let n denote zero or a positive integer such that

$$n-\mu\geq 0. \tag{29}$$

Omitting for the moment the case of equality and using (ref. 7, (2.4)) and (ref. 7. (2.2)), we obtain

$$R_0(x)=\frac{x^{-(1/2)\nu}}{\Gamma(n-\mu)}\frac{d^n}{dx^n}\int_0^x (x-y)^{n-\mu-1}y^{\mu+(1/2)\nu}f(y)dy \tag{30}$$

$$=x^{-(1/2)\nu}\frac{d^n}{dx^n}g(x;n-\mu), \tag{31}$$

$$R(\sqrt{x})=x^{1/2}\frac{d^n}{dx^n}g(x;n-\mu), \tag{32}$$

where $g(x;n-\mu)$ is the Riemann-Liouville fractional integral of order $n-\mu$.

The original unknown function Ψ in (1) and (2) is expressible in terms of ϕ, the unknown function in (26) as follows by (ref. 7, (3.3))

$$\Psi(x)=\frac{1}{2}x\psi\left(\frac{1}{4}x^2\right) \text{ and } \psi=S_{(1/2)\nu,-(1/2)\nu}h_1. \tag{33}$$

Then by (ref. 7, (2.5))

$$\psi(u)=u^{(1/2)\nu}\int_0^1 v^{(1/2)\nu}J_0(2\sqrt{uv})\,h_1(v)\,dv, \tag{34}$$

so

$$\Psi(x)=2^{-\nu}x^{\nu+1}\int_0^1 J_0(xy)\,\phi(y)\,dy, \tag{35}$$

a Hankel transform.

6. THE KERNEL

The kernel of (26)

$$K(x,y)=x\int_0^\infty t^{2\tau+1}J_0(x\sqrt{t^2+a^2})J_0(y\sqrt{t^2+a^2})\,dt$$
$$=2^\tau a^{\tau+1}D(x,y,a,-\tau-1).$$

Using Lemma 3 or

$$\int_0^\infty t^{2\tau+1}J_0(u\sqrt{t^2+a^2})\,dt=\frac{2^\tau\Gamma(\nu+1)}{u^{\nu+1}}a^{\tau+1}J_{-(\tau+1)}(ua)$$

(ref. 8, p. 417(5)).

This indicates the desired continuity on the function if we define

$$\begin{cases} K(0,y)=0, & y>0, \\ K(x,0)=\dfrac{2^\tau\Gamma(\nu+1)}{x^\nu}a^{\tau+1}J_{-(\tau+1)}(xa), & x>0. \end{cases} \tag{36}$$

If ν is an integer

$$\begin{cases} K(0,y)=0, & y\geq 0, \\ K(x,0)=(-1)^{\tau+1}\dfrac{2^\tau\Gamma(\tau+1)}{x^\tau}a^{\tau+1}J_{\tau+1}(xa), & x\geq 0, \end{cases} \tag{37}$$

then $0<x<1$ in (1) may be extended into $0\leq x\leq 1$.

Burlak[6] has considered the case $0\leq x\leq 1$ in (1), however, above results show that it will be valid under certain particular values of a and τ.