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Shell Structure of Two-Dimensional Nuclei

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The shell structure of atomic nuclei has an intimate relation with a geometrical property of the space. This property is examined further for a hypothetical two-dimensional nuclues. Two schemes of level sequence calculation, Nilsson scheme and a bounded harmonic oscillator potential scheme proposed by Sengupta and Ghosh, are adopted to determine the magic numbers. The determined numbers coincide exactly with those predicted by the particle capacity of circular rings; they are either 2, 6, 14, 22, 32, 44.... or 2, 6, 12, 22, 44..., depending on the magnitude λ of the spin-orbit interaction. The rule for the nuclear radius is not so pretty as the one for the three dimensional case. However, owing to this rule, a parameter of the potential well can be determined and the level sequence calculation becomes possible.

INTRODUCTION

IT has been shown in the previous report[1] that the shell structure of atomic nucleus has an intimate relation with a property of the space geometry. The magic numbers have been explained in terms of the particle capacity of spherical shells. The radii of these spherical shells form a simple sequence of integers and half -integers. This property will be examined in the present paper further for a hypothetical two-dimensional (2D) nucleus. The magic numbers for the 2D nuclei will be determined from the usual schemes for the filling of one-particle levels, and then will be compared with those predicted by the particle capacity of circular rings.

LEVEL SEQUENCE CALCULATIONS

The scheme for shell filling in the 2D nucleus can be determined by the same way as the 3D nucleus. It is well known that a central potential which is somewhat intermediate between a square well and a harmonic oscillator can give a reasonable level sequence if a spin-orbit interaction is added. The central scheme shown in Fig. 1 reveals this situation for 2D nuclei. For a test of numerological preciseness two schemes of level sequence calculation are attempted in the subsequent part.

(a) *The Nilsson Scheme*

Nilsson[2] found that the level sequence given by the expression

(1) J. L. Hwang. Chin. J. Phys. 2, 84(1964).
(2) S. G. Nilsson, Dan, Mat. Fys. Medd. 29, No. 16 (1955).

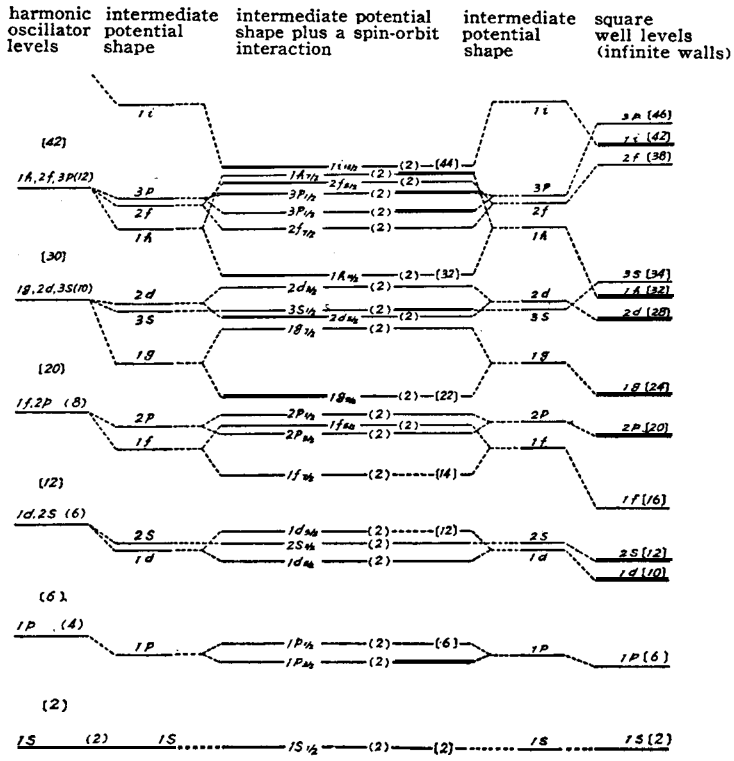

Fig. 1. The levels of an harmonic oscillator, a square well, an intermediate potential shape and an intermediate potential shape plus a spin-orbit interaction for a two-dimensional nucleus.

$$a_3(nlj) = \hbar\omega\left\{(\Lambda+\frac{3}{2}) - 2\kappa\langle \boldsymbol{l}\cdot\boldsymbol{s}\rangle_j - Dl(l+1)\right\}$$

can reproduce the experimental one very well. For the 2D case this expression is modified to be

$$a_2(nlj) = \hbar\omega\{(\Lambda+1) - 2\kappa\langle \boldsymbol{l}\cdot\boldsymbol{s}\rangle_j - Dl^2\},$$

where ω is the circular frequency of the oscillator, $\Lambda=2n$ + Z-2,

$$\langle \boldsymbol{l}\cdot\boldsymbol{s}\rangle_j = \frac{1}{2}(j^2 - s^2 - l^2) = \begin{cases} Z/2 & (j=l+\frac{1}{2}) \\ -Z/2 & (j=l-\frac{1}{2}), \end{cases}$$

and D is a number which takes the values

$D=0$ for $\Lambda=0, 1$
= -0.011 for **A = 2**
=0.0175 for if=3
= 0.020 for $\Lambda=4$.

$-2\kappa\hbar\omega_\Lambda$ represents the mean value of the spin-orbit interaction

$$\xi(r) = -\frac{\lambda\hbar^2}{4m^2c^2}\,\frac{1}{r}\,\frac{dV}{dr}$$

taken in the state (n, l), that is

$$2\kappa\hbar\omega_A = -\int_0^\infty \xi(r) R_{nl}^2(r) r^2 dr.$$

Since $V = \frac{1}{2} m\omega^2 r^2$, *we have*

$$\xi(r) = -\frac{\lambda\hbar^2\omega^2}{4mc^2} \text{ and } \kappa = \frac{\lambda\hbar\omega}{8mc^2}.$$

Here two parameters λ and ω are to be determined. λ is adjusted to be 250 which is approximately 5 times the value for 3D case. The well parameter ω can be determined from the *experimental nuclear radius* R by

$$R^2 = \frac{5}{3} \frac{\sum_\Lambda N_A \langle r^2 \rangle_{nl}}{\sum_\Lambda N_A} \text{ and } \langle r^2 \rangle_{nl} \frac{\hbar}{m\omega_A} = 2n + l - 1 = \Lambda + 1,$$

where $N_A = 2, 6, 12, 20\ldots$ respectively for $\Lambda = 0, 1, 2, 3, \ldots$ are the number of particles occupying the degenerate level of energy E_A. But the 2D nucleus is hypothetical and the radius R is *not obtainable* from the measurement. Nevertheless, as will be shown in the next section, by counting the particle capacity of the space, the radius for the closed shell nucleus with an isotropic harmonic oscillator potential well can be determined. The result is shown in Table 1. For nuclei of proton (neutron) number 22, 44 and 74, $\hbar\omega$ is determined to be 3.15, 2.11 and 1.59 respectively, and the level sequences are calculated for these nuclei. The shell formation of the energy levels can be seen readily from Fig. 2.

(b) Bounded Harmonic Oscillator Potentials

Sengupta and Ghosh[3] investigated a harmonic oscillator bounded by a potential wall of finite height;

$$V = -V_0 + \frac{1}{2} m\omega^2 r, \quad r < R$$
$$V = 0, \quad r > R.$$

The level sequence obtained agrees very well with the experimental one. A similar calculation for the 2D case is possible and is done here.

The radial part of the wave function is given by

$$R_{nl} = e^{-\rho/2} \rho^{(l/2)+(1/4)} {}_1F_1(-\lambda_{nl}, l+1, \rho),$$

where $\rho = \frac{\omega M r^2}{\hbar}$, $\lambda_{nl} = \frac{E_{nl}}{2\hbar\omega} - \frac{l+1}{2}$, ω is the circular frequency of the oscillator, and E_{nl} is the energy. ${}_1F_1$ is the confluent hypergeometric function. The rigid wall condition leads

(3) S. Sengupta and S. Ghosh, Phys. Rev. **115**, 1681 (1959).

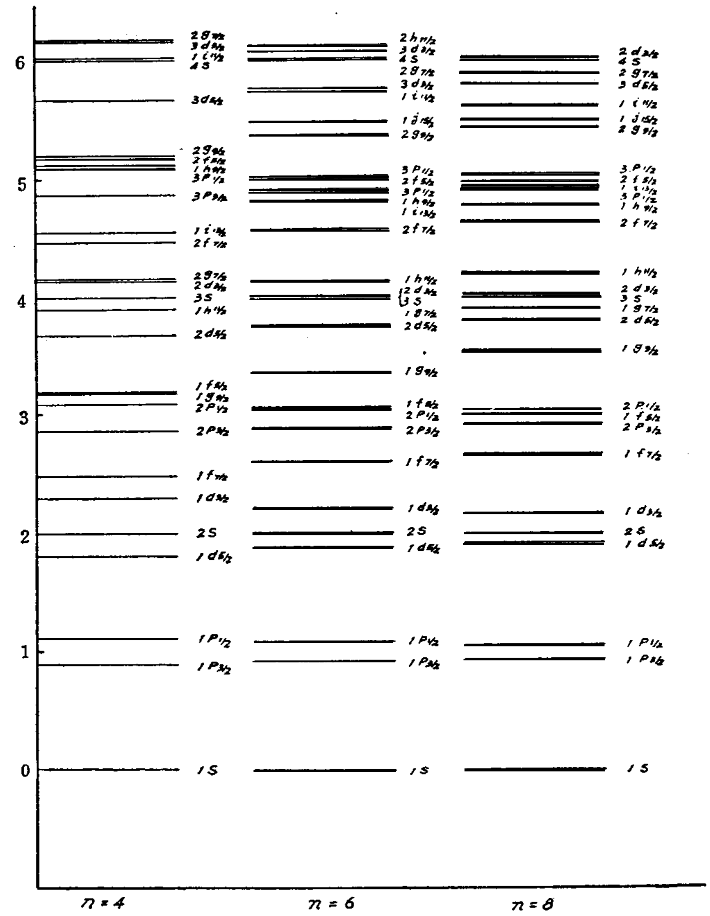

Fig. 2. The energy levels calculated by the Nilsson formula for the two-dimensional nuclei of neutron (proton) number 22, 44 and 74. The energy is expressed in units of $\hbar\omega$.

$$R_{nl}(\rho_0)=0, \text{ where } \rho_0=\omega MR^2/\hbar$$

and R is the nuclear radius. A suitable modification of the Tricomi expression for the zeros of hypergeometric function gives λ_{nl} for fixed values of ρ_0 and l as

$$\lambda_{nl}=\frac{l+j^2_{nl}}{4\rho_0}-\frac{(l+1)}{2}-\frac{(l+1)(l-1)}{6j^2_{nl}}-\frac{\rho_0}{12},$$

j^2_{nl} being the n-th zero of the Bessel function of order l. For not very large value of ρ_0 ($\rho_0<3n+l+3/2$), the energy levels are given by

$$E_{nl}=\frac{\hbar^2}{2MR^2}\left[j^2_{nl}+\frac{2(l+1)(l-1)}{3j^2_{nl}}\rho_0^2+\frac{\rho_0^2}{3}\right].$$

Using the Thomas-Frenkel expression and the same value of I, $\lambda=250$, as used in (a) the total spin-orbit energy can be calculated by the perturbation method, and it is given by

$$E_{s-o} = -\frac{\lambda\hbar^2}{2M^2C^2R^2}(\boldsymbol{l}\cdot\boldsymbol{s})\left[j_{nl}^2 - \frac{2(l+1)(l-1)}{3j_{nl}^2}\rho_0^2 + \frac{2}{3}\rho_0^2\right]\frac{\hbar^2}{2MR}$$

The same situation as arose in (a) arises here, that is, the values of circular frequency ω and the nuclear radius R remain undetermined as far as only the energetical consideration is made. They should be determined also from the result of the next section (Table 1). The level sequences for the same nuclei investigated in (a] are calculated and illustrated in Fig. 3.

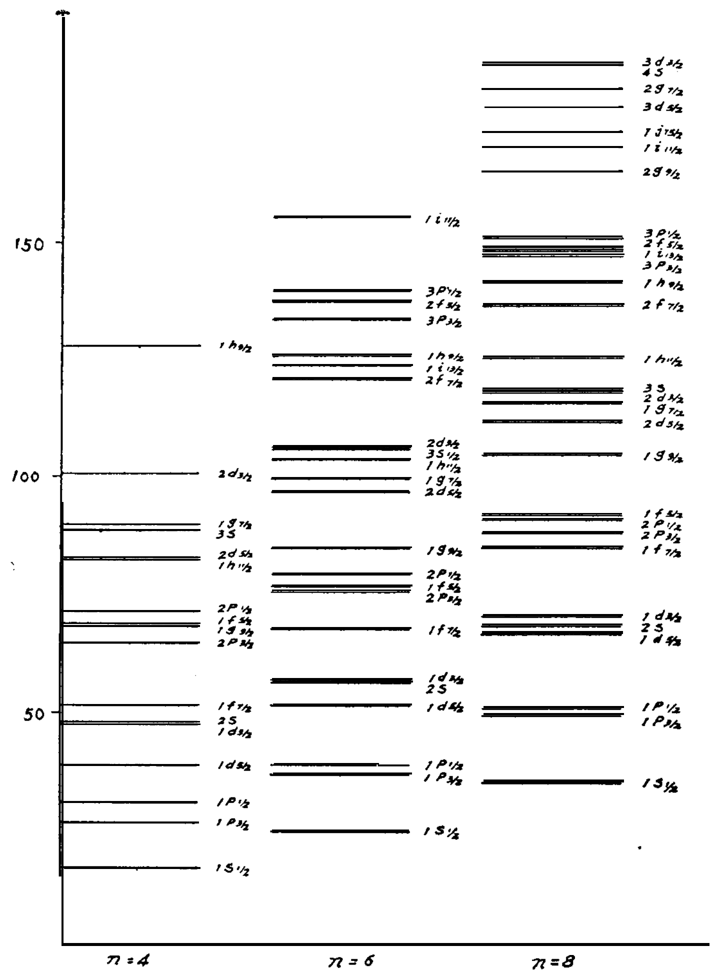

Fig. 3. The energy levels for bounded harmonic oscillators. The neutron (proton) numbers are 30, 56 and 90. The energy is expressed in units of $\hbar^2/2MR^2$.

On glancing Figs. 2 and 3, the tendency of shell formation of energy levels makes us incline to suggest that the magic numbers might be 2, 6, 12, 14, 22, 32.... with a straightforward analogy with the 3D case. However, as will be seen in the next section, we shall have to conclude that two particles occupying the 1 $f_{7/2}$ level cannot form an individual shell. The magic numbers should be either 2, 6, 14, 22, 32. ... or 2, 6, 12, 22, 32.. .., depending on the magnitude of spin-orbit interaction 1. If the $1f_{7/2}$ level is close to the lower shell, the former holds, and if it is close to the upper shell, the latter is true.

PARTICLE CAPACITY OF CIRCULAR RINGS

If the two dimensional nucleon (proton or neutron) is assumed to be a rigid circircle of radius r, then the number of circles N which can be inlaid into a circular ring of inner radius R_0-r and outer radius R_0+r can be calculated by an elementary geometry, and it is given by

$$N=\pi/\sin^{-1}(1/\eta) \quad \text{where } \eta=R_0/r.$$

Next we assign $r=1$ fm as did in the 3D case. Then whenever η takes a value listed in the first column of Table 1, N becomes an integer. These values of η do not make each ring overlap so much, and do not obstruct the circular motion of nucleons within the ring. Actually the number N is the same with the one obtained by arranging circles of the same radius over a plane as closely as possible but without overlapping with each other (Fig. 4).

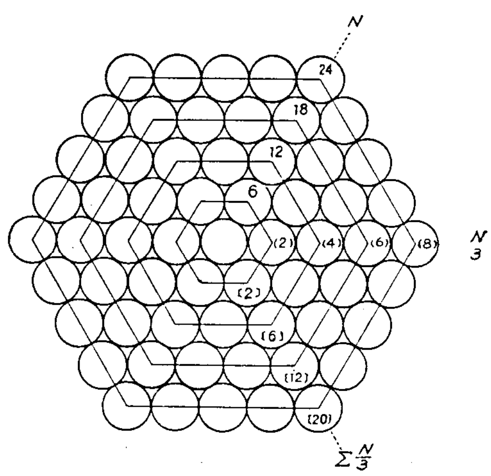

Fig. 4 The closest arrangement of circles on the plane. The arrangement of nucleons within the two-dimensional nucleus is generated from this pattern.

Referring Fig. 1 we find it reasonable to assume that the maximum number of protons and neutrons existing in a ring is N/3 for each kind and the remaining space of capacity N/3 is vaccant. Then the numbers of particles of one kind in the rings becomes 2, 4, 6, 8, 10.. .. as listed in the fourth column of Table 1, which are just equal to those of the particles occupying the major energetical shells of 2D nucleus with a harmonic oscillator potential. The aggregate of them gives the magic numbers 2, 6, 20, 30, 42.. ... Unlike the 3D nucleus, there is no evidence of discontinuity in the neighborhood of the shell (ring) corresponding to the $1f_{7/2}$ level.

Since the number of particles in the 1 $f_{7/2}$ level is too few to form an individual shell (ring) by itself, it is thought these two particles invade into either the third shell (ring) or the fourth shell (ring) and constitute something like a core within the nuleus. The sequencn of magic numbers for a *realistic* 2D nuclei is therefore 2, 6, 14, 22, 32. ... or 2, 6, 12, 22, 32.. .., which agrees with the prediction of the energy levels calculations made in the former section.

The rules for the 2D nuclear radius $R=\rho+1$ are then:

(1) *The closed shell 2D nuclei of proton (neutron) number 2,* 6, 14, *22,* 3.2, *44.... (or* 2, 6, *12,* 22, 32, 44. ....) *have the radii of proton (neutron) density 3.00, 4.86, 6.76, 8.66, 10.6, 12.5. ...fm respectively.*

Table 1. Number of circles N which can be inlaid into a circular ring of inner radius R_0-r and outer radius R_0+r: $N=\pi/\sin(1/\eta)$ and $\eta=R_0/r$. If we assume each ring contains N/3 neutrons, N/3 protons and N/3 vaccancies, then N/3 equals to the number of particles in a shell of the isotropic harmonic oscillator. For the two-dimensional nucleus, number of particles in each shell is modified as listed in the fifth or sixth columns. The nuclear radius is defined as $R=\eta+1$ fm.

η	$\Delta\eta$	N	Particles in each shell (ring)			Magic Numbers for 2D nucleus	
			Isotropic harmonic oscillator	2D nucleus			
2.00		6	2	2	2	2	2
3.86	1.86	12	4	4	4	6	6
5.76	1.90	18	6	(8)	6	14	12
7.66	1.90	24	8	8	(10)	22	22
9.57	1.91	30	10	10	10	32	32
11.47	1.90	36	12	12	12	44	44
13.38	1.91	42	14	14	14	58	58
15.29	1.91	48	16	16	16	74	74

(2) *The radii of proton (neutron) density of other 2D nuclei can be found by an interpolation between the above set of numbers 3.00, 4.86, 6.76, 8.66. ...fm.* These rules are not so beautiful as those of 3D case(4). However, without them we had no mean to know the magnitude of the potential well parameters and could not determine the level sequence. We have thus learned that in the case of 2D nucleus the shell structure is also intimately related with a geometrical property of the space.

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(4) J. L. Hwang, Chin. J. Phys. 1. 24 (1963).