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Generators for the Liénard-Wiechert Electromagnetic Field

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Using Newman-Unti coordinates in Minkowski space, we obtain the generators of several quantities associated with the Liénard-Wiechert field.

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I. Introduction

The motion of a classical charged particle e in Minkowski space is an active theme of research [1, 2] because we have not yet an answer –without ambiguities– to the question: which equation of motion correctly describes the interaction of e with its proper electromagnetic field? Most people accept the Lorentz-Dirac equation (LDE) [3-11]. However, this equation has anomalies like run-away solutions, a violation of causality, asymptotic boundary conditions and the non-Newtonian character of the initial data [4, 8, 9, 12, 13]. The LDE has been deduced in a number of ways, for example, by employing mass renormalization [3, 14], ad hoc hypothesis [15-19], arbitrary modifications [20, 21] of the Maxwell tensor T_{ab} for the Liénard-Wiechert (LW) field. In our opinion, before obtaining any equation of motion for a charge e we must learn much more about the structure of the LW solution [4, 5, 22]. We believe that the superpotential concept could give us new information about the electromagnetic field emitted by the point charge. Freud [23] created the idea of a superpotential when he deduced a generator for the Einstein energy-momentum canonical pseudotensor [24] in general relativity. This idea means that the annulation of the divergence of "something" implies the existence of a superpotential generating it.

On the other hand, the attempt to understand the dynamics of the charge leads, in a natural way, to the use of Newman-Unti (NU) coordinates [25-28]. This occurs because in the construction of this coordinate system the motion of the particle under study takes pat. We show this in Sec. 2, and in Sec. 3 we obtain the superpotentials for the 4-potential A_c and the Faraday tensor F_{ab} [4, 5, 24, 29] associated to LW field. Sec. 4 expresses the generators for the bounded $\underset{B}{T_{ac}}$ and radiative $\underset{R}{T_{ac}}$ parts of the Teitelboim's splitting [30, 31] for the Maxwell tensor T_{ac}.

II. NU coordinates

Newman-Unti [25] constructed a coordinate system appropriate for the analysis of fields depending on retarded effects over the path $q^c(u)$ of the charged particle. The NU coordinates may be generalized to asymptotically flat curved spaces, there by giving a new method [15-18] for the problem of motion in general relativity.

In Minkowski space, the coordinates (x, y, z, t) with line element

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 \tag{1}$$

are usual. This metric is a very simple one, but some objects have complicated expressions. Newman-Unti's essential idea is to employ new coordinates named (x^1, x^2, r, u) which allow are to introduce simplifications into the LW electromagnetic tensors, although the simplicity of Eq. (1) is lost. For its construction we fix an arbitrary world line $q^c(u)$ traced by the point charge; u is taken to be the corresponding proper time and the relationship between Minkowskian and NU coordinates is given by [25-28] $(i = \sqrt{-1})$:

$$\begin{aligned} x &= q^1(u) + \frac{r(\eta+\bar{\eta})}{2\sqrt{2}P}, \quad y = q^2(u) + \frac{ir(\bar{\eta}-\eta)}{2\sqrt{2}P}, \\ z &= q^3(u) + \frac{r(\eta\bar{\eta}-1)}{2\sqrt{2}P}, \quad t = q^4(u) + \frac{r(\eta\bar{\eta}+1)}{2\sqrt{2}P}, \\ 2\sqrt{2}P &= (\dot{q}^4 + \dot{q}^3) + (\dot{q}^4 - \dot{q}^3)\eta\bar{\eta} - (\dot{q}^1 - i\dot{q}^2)\eta - (\dot{q}^1 + i\dot{q}^2)\bar{\eta}, \end{aligned} \tag{2}$$

where $\eta \equiv x^1 + ix^2$, the point over a quantities means $\partial/\partial u$, a bar denotes the complex conjugate quantity, and r is the retarded distance [4, 20, 24, 31-34] from point (x, y, z, t) to the path $q^c(u)$. We see that the function P contains the velocity $\dot{q}^C$, this means that NU coordinates intrinsically have information about the motion of the particle.

If we substitute (2) into (1) the metric

$$ds^2 = g_{ab}dx^a dx^b = \frac{r^2}{2P^2}d\eta d\bar{\eta} - 2drdu - \left(1 - \frac{2\dot{P}}{P}r\right)du^2 \tag{3}$$

follows. This metric (3) is more complicated than (1), however, we next show that (3) introduces remarkable simplifications in the expressions of the electromagnetic field radiated by the point particle. We note that $\dot{P}$ appears in (3), showing that the acceleration $\ddot{q}^C$ participates in the construction of the NU metric. Now we give some useful relations employed in the search for superpotentials. The non-zero NU Christoffel symbols are:

$$\begin{aligned} &\Gamma^1{}_{11} = -\Gamma^1{}_{22} = \Gamma^2{}_{12} = -\frac{1}{P}\frac{\partial P}{\partial x^1}, \\ &\Gamma^1{}_{13} = +\Gamma^2{}_{23} = \frac{1}{r}, \\ &\Gamma^1{}_{12} = -\Gamma^2{}_{11} = \Gamma^2{}_{22} = -\frac{1}{P}\frac{\partial P}{\partial x^2}, \\ &\Gamma^4{}_{11} = \Gamma^4{}_{22} = \frac{r}{2P^2}, \end{aligned}$$

$$\begin{aligned}
&\Gamma^2{}_{24} = +\Gamma^1{}_{14} = -\Gamma^4{}_{44} = \Gamma^3{}_{34} = -\frac{\dot{P}}{P},\\
&\Gamma^3{}_{j4} = -r\frac{\partial}{\partial x^j}\left(\frac{\dot{P}}{P}\right),\\
&\Gamma^j{}_{44} = -\frac{2P^2}{r}\frac{\partial}{\partial x^j}\left(\frac{\dot{P}}{P}\right), \quad j = 1,2\\
&\Gamma^3{}_{11} = \Gamma^3{}_{22} = -\frac{r}{2P^2}\left(1 - r\frac{\dot{P}}{P}\right),\\
&\Gamma^3{}_{44} = 3r\left(\frac{\dot{P}}{P}\right)^2 - r\frac{\ddot{P}}{P} - \frac{\dot{P}}{P}.
\end{aligned} \tag{4}$$

It is important to observe that $\Gamma^3{}_{44}$ depends on $\ddot{P}$, that is, on the superacceleration [20] $\dddot{q}^C$, which has great relevance in the LDE [3-11, 14, 15-21, 24] where the radiation reaction effect [4, 9, 12, 24, 35] is taken into account.

Also, as the curvature tensor [24, 36] generated by (4) must be zero in special relativity, we obtain the following identities for P [28]:

$$\frac{\partial}{\partial \eta}\left[P^2\frac{\partial}{\partial \eta}\left(\frac{\dot{P}}{P}\right)\right] = 0, \quad \frac{\partial^2}{\partial\eta\partial\bar{\eta}}\ln P = \frac{1}{8P^2}, \quad 4P^2\frac{\partial^2}{\partial\eta\partial\bar{\eta}}\left(\frac{\dot{P}}{P}\right) + \frac{\dot{P}}{P} = 0, \tag{5}$$

useful in the deduction of superpotentials.

It is a complicated task to write each Cartesian component of the LW field, but the use of NU coordinates allows are to express the 4-potential A^c and the Faraday antisymmetric tensor F_{ab} in a simple form [25, 37]:

$$(A^C) = e\left(0, 0, \frac{\dot{P}}{P}, \frac{1}{r}\right), \tag{6}$$

and $F_{AC} = 0$, with the exception of

$$F^{34} = -\frac{e}{r^2}, \quad F^{j3} = -\frac{2eP^2}{r^2}\frac{\partial}{\partial x^j}\left(\frac{\dot{P}}{P}\right), \quad j = 1,2, \tag{7}$$

which satisfy the Lorenz condition; (denotes covariant derivative [24, 36])

$$A^c{}_{;c} = 0 \tag{8}$$

and the Maxwell equations in vacuum:

$$F^{ab}{}_{;a} = 0. \tag{9}$$

From (7) is clear that when $\ddot{q}^C = 0$ (then $\dot{P} = 0$) the Faraday tensor is of Coulomb type in NU coordinates.

III. Superpotentials for A^c and F^{ab}

When we speak about a potential, we think of a tensorial object that under differentiation gives rise to another field tensor verifying a conservation law (continuity equation); so, we believe that such a potential is close to the fundamental structure of the corresponding physical field, LW in our case. We seek for an antisymmetric tensor W^{bc} with the property

$$A^b = W^{bc}{}_{;c} \tag{10}$$

such that Eq. (8) follows immediately. If we evaluate (10) at $b = 1, \ldots, 4$ then, by use of the definition of covariant derivative and equations (4) and (6), there results a set of equations which admit the non-unique solution:

$$W^{bc} = 0, \quad \text{with the exception of} \quad W^{34} = -W^{43} = -\frac{e}{2}. \tag{11}$$

Putting (11) into (10) gives (6), which means that W^{ab} generates the 4-potential A^c. If, on the other hand, $\nu^c = \dot{q}^c$ is the 4-velocity of the charge and k^c is the null vector joining the point (x^1, x^2, r, u) with its associated retarded event over the world line, then it is easy to see that in NU coordinates

$$(\nu^c) = \left(0, 0, \frac{\dot{P}}{P} r, 1\right), \quad (k^c) = (0, 0, r, 0). \tag{12}$$

Therefore (11) can be written in a tensorial form valid for any coordinate system:

$$W^{bc} = \frac{e}{2r}(\nu^b k^c - \nu^c k^b), \tag{13}$$

in accordance with (34) of [34].

For the Faraday tensor we must find a superpotential $\underset{F}{K}^{abc} = -\underset{F}{K}^{bac} = -\underset{F}{K}^{acb}$ (which shall guarantee the antisymmetry of F^{ab}) with the property

$$F^{ab} = \underset{F}{K}^{acb}{}_{;c}, \tag{14}$$

thus implying the Maxwell equations (9) as identities. Equations (4), (5), (7) and (14) lead to a differential equation system which has the solution $\underset{F}{K}^{abc} = 0$ with the exception of the components

$$\underset{F}{K}^{34j} = \underset{F}{K}^{j34} = -\underset{F}{K}^{j43} = e\frac{\partial}{\partial x^j}\left(\frac{P}{r}\right)^2, \quad j = 1, 2. \tag{15}$$

In [34] we found a superpotential $\underset{F}{K}^{abc}$, in Minkowskian coordinates, with a non-local character – because the expression (35) in Ref. [34] contains integrals over the past history of the particle; in this sense our relation (15) is simpler than the previous one.

Plebañski [22] and Teitelboim [38] introduced the method of average field [39] as an alternative to the process used in [3] and thus obtained, in a simple form, the LDE in the absence of gravity. This method offers easier calculations than the method of Dirac because it can be applied with a Bhabha [40]-Synge [20] cylinder and because it does not need T_{ab} but only F_{ab}. So, in order to use the Plebañski-Teitelboim technique it is necessary to compute fluxes of F_{ij} through hypersurfaces. It is in these computations where the usefulness of the superpotential $\underset{F}{K}^{abc}$ is evident.

IV. Superpotentials for the bounded and radiative parts of T_{ab}.

The Maxwell tensor T_{ij} associated to the LW field admits the Teitelboim splitting [30]

$$T^{ab} = \underset{R}{T}^{ab} + \underset{B}{T}^{ab}, \tag{16}$$

where $\underset{B}{T}^{ij}$ and $\underset{R}{T}^{ij}$ are the bounded and radiative parts, respectively. They are dynamically independent because they separately satisfy–outside of the world line of the charge– the conservation laws:

$$\underset{B}{T}^{ab}{}_{;a} = 0, \tag{17}$$

$$\underset{R}{T}^{ab}{}_{;a} = 0, \tag{18}$$

Weert [41] proved, in Minkowskian coordinates, that from (17) there follows the existence of a superpotential $\underset{B}{K}^{ijc} = -\underset{B}{K}^{jic}$, $\underset{B}{K}^{ijc} + \underset{B}{K}^{jci} + \underset{B}{K}^{cij} = 0$, $\underset{B}{K}^{ijc}{}_{;c} = 0$ [which shall guarantees the symmetry of $\underset{B}{T}^{ij}$], such that:

$$\underset{B}{T}^{ij} = \underset{B}{K}^{icj}{}_{;c}, \tag{19}$$

but he did not show the deduction of his generator. The method employed at (14) may now be used to determine $\underset{B}{K}^{abc}$ in NU coordinates. In fact, the non-zero NU components of $\underset{B}{T}^{ij}$ are [27]:

$$\begin{aligned} &\underset{B}{T}^{11} = \underset{B}{T}^{22} = e^2 \frac{P^2}{r^6}, \quad \underset{B}{T}^{j3} = -\frac{2e^2P^2}{r^4}\frac{\partial}{\partial x^j}\left(\frac{\dot{P}}{P}\right), \quad j = 1, 2 \\ &\underset{B}{T}^{33} = -\frac{e^2}{2r^2}\left(1 - 2r\frac{\dot{P}}{P}\right), \quad \underset{B}{T}^{34} = \frac{e^2}{2r^4}. \end{aligned} \tag{20}$$

Then (4), (5) and (19) allow the solution $\underset{B}{K}^{abc} = 0$, except for

$$\begin{aligned} &\underset{B}{K}^{j33} = \frac{2e^2P^2}{r^3}\frac{\partial}{\partial x^j}\left(\frac{\dot{P}}{P}\right), \quad j = 1, 2 \\ &\underset{B}{K}^{131} = \underset{B}{K}^{232} = \frac{e^2P^2}{2r^5}, \quad \underset{B}{K}^{343} = \frac{e^2}{2r^3}, \end{aligned} \tag{21}$$

which is equivalent to Weert's result; this author also showed [41] that (19) simplifies the computation of the $\underset{B}{T}_{ac}$ fluxes through a 3-surface around the charge. Weert did not discover the physical meaning of his superpotential, but it was done later [42], where it was found that $\underset{B}{K}_{abc}$ is the intrinsic angular momentum density [43] of the LW field. This important result was possible due to the fact that the Weert generator has [31, 34, 42, 44-47] exactly the same symmetries – which guarantee that $\underset{B}{T}^{ij}$ is symmetric – as the Lanczos spintensor [48-54]:

$$\begin{aligned} &\underset{B}{K}_{abc} = -\underset{B}{K}_{bac}, \quad \underset{B}{K}_{a}{}^{c}{}_{c} = 0, \\ &\underset{B}{K}_{abc} + \underset{B}{K}_{bca} + \underset{B}{K}_{cab} = 0, \quad \underset{B}{K}_{ab}{}^{c}{}_{;c} = 0. \end{aligned} \tag{22}$$

The properties (22) lead to various results: a Petrov classification [44] for the LW field, the splitting [47] of $\underset{B}{K}_{abc}$ as proposed by López [55], and a possible relationship [56-60] between the Lanczos potential and the gravitational angular momentum in general relativity.

Similarly, the continuity equation (18) implies that the radiative part [61] is generated by a superpotential $\underset{R}{K}^{ijc} = -\underset{R}{K}^{jic}$ via the expression:

$$\underset{R}{T}^{ij} = \underset{R}{K}^{icj}{}_{;c}. \tag{23}$$

The NU components of $\underset{R}{T}^{ij}$ are given by [28] $\underset{R}{T}^{ab} = 0$, except for

$$\underset{R}{T}^{33} = -\frac{8e^2P^2}{r^2}\frac{\partial}{\partial\eta}\left(\frac{\dot{P}}{P}\right)\cdot\frac{\partial}{\partial\bar{\eta}}\left(\frac{\dot{P}}{P}\right). \tag{24}$$

With (4), (23) and (24) we obtain a non-trivial system of coupled partial differential equations which we will not write here. We limit ourselves to present a solution, showing only the terms different from zero (b = 1, 2):

$$\begin{aligned} &\underset{R}{K}^{b34} = \frac{e^2P\dot{P}}{2r^2}\frac{\partial}{\partial x^b}\left(\frac{\dot{P}}{P}\right), \\ &\underset{R}{K}^{132} = \underset{R}{K}^{231} = \frac{-e^2P^4}{r^3}\frac{\partial}{\partial x^1}\left(\frac{\dot{P}}{P}\right)\cdot\frac{\partial}{\partial x^2}\left(\frac{\dot{P}}{P}\right), \\ &\underset{R}{K}^{b33} = \frac{-e^2P\dot{P}}{2r^2}\left(5 - r\frac{\dot{P}}{P}\right)\frac{\partial}{\partial x^b}\left(\frac{\dot{P}}{P}\right), \\ &\underset{R}{K}^{b3b} = \frac{e^2}{2r^3}\left[\dot{P}^2 - 2P^4\left(\frac{\partial}{\partial x^b}\left(\frac{\dot{P}}{P}\right)\right)^2\right], \\ &\underset{R}{K}^{343} = \frac{2e^2P^3}{r^3}\int_0^u \frac{\dot{P}^2}{P^5}du, \end{aligned} \tag{25}$$

where the integration is made keeping x^1 and x^2 (or η and $\bar{\eta}$) constant. With (25) it is possible to verify the property $(\underset{R}{K}^{cij} + \underset{R}{K}^{cji})_{;c} = 0$ which guarantees the symmetry of $\underset{R}{T}^{ij}$. We note that all components of $\underset{R}{K}^{abc}$ involve $\dot{P}$, which is not surprisingly because the radiated field is related [4, 5, 62, 63] to the acceleration of the charge. The non-local character of the radiative superpotential has been confined to the quantity $\underset{R}{K}^{343}$, which is the only component involving an integral over the world line: This integral appears because the process of measuring the radiation rate is intrinsically non-local [64, 65]. In the electrodynamics of classical charged particles in curved spaces [39, 66-68] there are also non-local quantities. In [27, 31, 33, 34, 45, 46, 61, 69, 70] the Minkowskian coordinates and the Fermi triad [36, 71] are employed to determine another non-local radiative superpotentials for $\underset{R}{T}^{ij}$, however, the physical meaning of $\underset{R}{K}^{abc}$ is not yet known. It is interesting to note that, the procedure employed to obtain (25) is also useful for constructing [54, 72] Lanczos spintensors for the Robinson-Trautman metrics.

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