Non-Static Spherically Symmetric Perfect Fluid Solutions

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We investigate solutions of the Einstein field equations for the case of a non-static spherically symmetric perfect fluid using different equations of state. The properties of some exact spherically symmetric perfect fluid solutions which contain shear are obtained. We obtain three different solutions; out of these one turns out to be an incoherent dust solution and the other two are stiff matter solutions.

PACS. 04.20.jb – Exact solutions.

I. Introduction

There is no shortage of exact solutions of the Einstein field equations (EFEs). However, because General Relativity is highly non-linear, it is not always easy to understand what qualitative features solutions might possess. Different people have been working on the investigation of spherically symmetric perfect fluid solutions with shear [1-5]. Nearly all of these solutions have been obtained by imposing symmetry conditions. It is known that non-linear partial differential equations admit large classes of solutions, many of which are unphysical.

The EFEs for a static spherically symmetric distribution of perfect fluid have been investigated by many authors using different approaches [6]. One approach is to prescribe an equation of state, $\rho = \rho(p)$ which relates the energy density ρ and isotropic pressure p. In this paper we shall extend this idea of solving the EFEs to non-static spherically symmetric spacetimes. We shall examine systematically the field equations for the non-static spherically symmetric perfect fluid solutions. We obtain three different solutions which have non-zero shear. Since most of the models in the literature [6] for spherically symmetry are shear-free, it will be interesting to study non-static solutions which contain shear.

The breakup of the paper is as follows. In section II we shall write down the field equations. In the third section we attempt to find possible solutions in three classes using different equations of state. In section IV we shall evaluate kinematic quantities for the solutions obtained. Finally in the last section we shall conclude our discussion.

II. Field equations

The non-static spherically symmetric metric has the form [5]

$$ds^2 = e^{2\nu(r,t)}dt^2 - e^{2\lambda(r,t)}dr^2 - R^2(r,t)d\ \ ^2, \qquad (1)$$

where $d\;\;{}^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$.

For a perfect fluid distribution, the energy-momentum tensor is given by

$$T_{ab} = (\rho + p)u_a u_b - p g_{ab}, \quad u_a u^b = 1, \quad a, b = 0, 1, 2, 3. \tag{2}$$

where ρ and p are the energy density and the pressure of the fluid, respectively. The four velocity of the fluid has the form $u^a = (e^{-\nu(r,t)}, 0, 0, 0)$. The field equations for the metric (1) can be written down [6]

$$\square\rho = \frac{1}{R^2} - \frac{2}{R}e^{-2\lambda}\Big(R'' - R'\lambda' + \frac{R'^2}{2R}\Big) + \frac{2}{R}e^{-2\nu}\Big(\dot{R}\dot{\lambda} + \frac{\dot{R}^2}{2R}\Big), \tag{3}$$

$$\square p = -\frac{1}{R^2} + \frac{2}{R}e^{-2\lambda}\Big(R'\nu' + \frac{R'^2}{2R}\Big) - \frac{2}{R}e^{-2\nu}\Big(\ddot{R} - \dot{R}\dot{\nu} + \frac{\dot{R}^2}{2R}\Big), \tag{4}$$

$$\square pR = e^{-2\lambda}\big((\nu'' + \nu'^2 - \nu'\lambda')R + R'' + R'\nu' - R'\lambda'\big)$$

$$-e^{-2\nu}((\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu})R + \ddot{R} + \dot{R}\dot{\lambda} - \dot{R}\dot{\nu}), \tag{5}$$

$$\dot{R}' - \dot{R}\nu' - R'\dot{\lambda} = 0, \tag{6}$$

where the dot denotes partial derivative with respect to time 't' and the prime indicates partial derivative with respect to the coordinate 'r'. The spatial coordinate 'r' refers to the comoving radius and '□' is the gravitational constant. The consequences of energy-momentum conservation $T^{ab}_{;b} = 0$ are the relations

$$p' = -(\rho + p)\nu', \quad \dot{\rho} = -(\rho + p)\Big(\dot{\lambda} + 2\frac{\dot{R}}{R}\Big). \tag{7}$$

We now consider the equation of state

$$p = (\gamma - 1)\rho, \quad \rho + p = \rho\gamma, \quad 1 \quad \gamma \quad 2, \tag{8}$$

where γ is a constant. (The limits on γ result from the requirement that the stresses be pressures rather than tensions and that the speed of sound in the fluid be less than the speed of light in vacuum). For $\gamma = 1$, the pressure vanishes, so that the equation of state is that of incoherent dust. For $\gamma = \frac{4}{3}$, the equation of state is that of a photon gas or a gas of non-interacting relativistic particles. For $\gamma = 2$, the equation of state reduces to the stiff matter case.

III. Non-static spherically symmetric perfect fluid solutions

To simplify the field equations we solve them for some special cases:

III-1. The case $R = R(t)$, $\lambda = \lambda(\mathbf{t})$

This class is identical with the well-known Kantowski–Sachs class of cosmological models [7-9]. We can choose $R = t$ without loss of generality, use the equation of state given by Eq. (8) and try to find all of the possible solutions.

When $\gamma = 1$

This gives $p = 0$, the EFEs. (3-6) will reduce to

$$\square\rho = \frac{1}{t^2} + \frac{2}{t}e^{-2\nu}\Big(\dot{\lambda} + \frac{1}{2t}\Big), \tag{9}$$

$$0 = 2t\,\dot{\nu}\,e^{-2\nu} - 1 - e^{-2\nu}, \tag{10}$$

$$0 = -e^{-2\nu}[(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu})t + \dot{\lambda} - \dot{\nu}]. \tag{11}$$

Eq. (10) can easily be solved to give

$$\nu = \frac{1}{2}\ln\left|\frac{t}{c-t}\right|,$$

where c is an arbitrary constant. For this value of ν, Eq. (9) gives

$$\dot{\lambda} = \frac{\rho\square t^3 - c}{2t(c-t)}.$$

Substituting the values of $\ddot{\lambda}, \dot{\lambda}^2, \dot{\lambda}, \dot{\nu}$ in Eq. (11), we have

$$\dot{\rho} + \frac{3c-4t}{2t(c-t)}\rho = \frac{t^2\square}{2(t-c)}\rho^2.$$

Solving this, we get

$$\rho = \left[\square\left\{\left(\frac{t}{c-t}\right)^{\frac{1}{2}} - \sin^{-1}\left(\frac{t}{c}\right)^{\frac{1}{2}}\right\}\sqrt{ct^3 - t^4} + c_1\sqrt{ct^3 - t^4}\right]^{-1}, \tag{12}$$

where c_1 is an integration constant. By substituting this value of ρ in $\dot{\lambda}$ and integrating, the value of λ becomes

$$\lambda = \ln\left[c_2\left\{1 - \left(\frac{c-t}{t}\right)^{\frac{1}{2}}\sin^{-1}\left(\frac{t}{c}\right)^{\frac{1}{2}} + \left(\frac{c-t}{t}\right)^{\frac{1}{2}}\frac{c_1}{\square}\right\}\right], \tag{13}$$

where c_2 is an integration constant. The energy-momentum conservation relations (7) are also satisfied by this solution.

The resulting spacetime is

$$ds^2 = \frac{t}{c-t}dt^2 - \left[c_2\left\{1 - \left(\frac{c-t}{t}\right)^{\frac{1}{2}}\sin^{-1}\left(\frac{t}{c}\right)^{\frac{1}{2}} + \left(\frac{c-t}{t}\right)^{\frac{1}{2}}\frac{c_1}{\square}\right\}\right]^2 dr^2 - t^2 d\ \ ^2. \tag{14}$$

If we take the special case for which $\lambda = constant$ and $\gamma = 2(\rho = p)$, then the EFEs (3-6) will give

$$\nu = \ln\left|\frac{\alpha^2 t^2}{1 - \alpha^2 t^2}\right|^{\frac{1}{2}},$$

where α is an arbitrary constant. The energy density and the pressure can be evaluated as

$$\rho = p = \frac{1}{\square\alpha^2 t^4}. \tag{15}$$

The corresponding metric will be

$$ds^2 = \frac{\alpha^2 t^2}{1-\alpha^2 t^2} dt^2 - dr^2 - t^2 d\ \ ^2. \tag{16}$$

III-2. The case $R = R(t)$**,** $\lambda = \lambda(r,t)$

This class of solution was examined by Korkina and Martinenko [10, 11]. In this case the EFEs will be the same as Eqs. (3-6) except that now $\lambda = \lambda(r,t)$. We solve this system of partial differential equations using Eq. (8).

When $\gamma = 2$

In this case $p = \rho$, the EFEs (3-6) will give

$$\rho = \frac{1}{\square t^2} \square\, y\left(1 + \frac{2\dot{z}}{z} t\right) + 1\Big], \tag{17}$$

$$p = -\frac{1}{\square t^2}[\dot{y}t + 1 + y], \tag{18}$$

$$p = -\frac{1}{\square t^2} \square\, y\frac{\ddot{z}}{z}t^2 + \dot{y}\frac{\dot{z}}{2z}t^2 + y\frac{\dot{z}}{z}t + \frac{\dot{y}}{2}t\Big], \tag{19}$$

where $y = e^{-2\nu(t)}$, $z = e^{\lambda(r,t)}$. From Eqs. (18) and (19), we have

$$\ddot{z} + \dot{z} \square \frac{\dot{y}}{2y} + \frac{1}{t}\Big] - z \square \frac{\dot{y}}{2yt} + \frac{y+1}{yt^2}\Big] = 0. \tag{20}$$

This second order non-linear partial differential equation can be solved using Herlt's method [5] by choosing

$$y(t) = e^{-2\nu} = \frac{1}{n^2-1} + \beta t^{-2(n+1)}, \quad n^2 \neq 1 \tag{21}$$

where β is an arbitrary constant. Eq. (20) then becomes

$$\ddot{z} + A\dot{z} - Bz = 0, \tag{22}$$

where

$$A = \frac{t^{-1} - n(n^2-1)\beta t^{-2n-3}}{1 + \beta(n^2-1)t^{-2n-2}}, \tag{23}$$

$$B = \frac{n^2 t^{-2} - n(n^2-1)\beta t^{-2n-4}}{1 + \beta(n^2-1)t^{-2n-2}}. \tag{24}$$

Eq. (22) has the special solution $z_s = t^n$. Let us substitute

$$z = C(r,t)\,t^n \tag{25}$$

into Eq. (22), we obtain

$$\ddot{C} + [A + 2nt^{-1}]\dot{C} = 0. \tag{26}$$

This can easily be solved; the general solution becomes

$$z(r,t) = t^n \left\{ \beta_1(r) \int_{\beta_2(r)}^{t} \frac{t'^{-n}}{\sqrt{t'^{2n+2} + \beta(n^2-1)}} dt' \right\}, \tag{27}$$

where $\beta_1(r)$ and $\beta_2(r)$ are arbitrary functions of the variable r. The energy density ρ and pressure p can now be computed easily using Eqs. (17) and (18).

The corresponding metric will become

$$ds^2 = [y(t)]^{-1} dt^2 - z^2(r,t) dr^2 - t^2 d\ \ ^2, \tag{28}$$

where $y(t)$ and $z(r,t)$ are given by Eqs. (21) and (27) respectively. It is to be noticed that this solution corresponds to the solution obtained by Herlt [5].

III-3. The general case $R = R(r,t)$

To solve this general case we consider the following two assumptions:

$$(i)\nu = 0, \lambda = \lambda(r,t); \quad (ii)\nu = \nu(r,t), \lambda = 0.$$

(i) When $\nu = 0, \lambda = \lambda(r,t)$

The EFE (6) implies that $\lambda = \ln|fR'|$, where $f(r)$ is an arbitrary function of r and R is an arbitrary function of coordinates r and t. Also $R' \neq 0$, which implies that $R \neq R(t)$.

Here two cases arise i.e. either $R = R(r)$ or $R = R(r,t)$. For $R = R(r)$, we can have ρ and p by replacing λ in Eqs. (3) and (4) respectively.

$$\rho = \frac{1}{\Box}\left(\frac{1}{R^2} - \frac{1}{f^2R^2} + \frac{2f'}{PR'f^3}\right),$$

$$p = \frac{1}{\Box}\left(-\frac{1}{R^2} + \frac{1}{f^2R^2}\right).$$

But Eq. (5) gives

$$p = -\frac{f'}{\Box RR'f^3}.$$

By comparing the two values of p, we obtain

$$R = \frac{\sqrt{f^2-1}}{lf},$$

where l is an integration constant. The resulting metric becomes

$$ds^2 = dt^2 - \frac{f'^2}{l^2 f^2(f^2-1)}dr^2 - \frac{f^2-1}{l^2 f^2}d\ \ ^2. \tag{29}$$

This turns out to be a class of spherically symmetric static spacetimes. For $f = \frac{1}{\sqrt{1-r^2}}$ and $l = 1$, it reduces to the Einstein metric. From the above equations we obtain $\rho = \frac{3l^2}{\square}$, $p = -\frac{l^2}{\square}$ which implies that $\rho + 3p = 0$.

(ii) When $\nu = \nu(r,t), \lambda = 0$, the EFE (6) will reduce to $\nu = \ln|g\dot{R}|$, where $g(t)$ is an arbitrary function of t and R is an arbitrary function of the coordinates r and t. Also $\dot{R} \neq 0$, which implies that $R \neq R(r)$. This means that either $R = R(t)$, or $R = R(r,t)$. For $R = R(t)$, we have a spacetime similar to Eq. (16). The quantities ρ and p also turn out to be the same.

For $R = R(r,t)$, the solutions need to be investigated.

IV. Kinematics of the velocity field

The spherically symmetric solutions can be classified according to their kinematical properties [6]. The rotation is given by

$$\omega_{ab} = u_{[a;b]} + \dot{u}_{[a}u_{b]}. \tag{30}$$

The acceleration can be found by

$$\dot{u}_a = u_{a;b}u^b. \tag{31}$$

For the expansion we have

$$\Theta = u^a_{;a}. \tag{32}$$

The components of the shear-tensor are given by

$$\sigma_{ab} = u_{(a;b)} + \dot{u}_{(a}u_{b)} - \frac{1}{3}\Theta h_{ab}, \tag{33}$$

where $h_{ab} = g_{ab} - u_a u_b$ is the projection operator. The square brackets denote antisymmetrization and the round brackets indicate symmetrization. The shear invariant is given as $\sigma_{ab}\sigma^{ab}$.

Now we find all the above quantities for the solutions obtained. The rotation and the acceleration are zero for all the solutions. The expansion, for the first solution, is

$$\Theta = \left(\frac{(c-t)}{t}\right)^{\frac{1}{2}} \square \frac{\rho\square t^3 - c}{2t(c-t)} + \frac{2}{t}\Bigg]. \tag{34}$$

The components of the shear-tensor are given by

$$\sigma_{11} = \frac{2}{3}\left(\frac{c-t}{t}\right)^{\frac{1}{2}} \square \frac{1}{t} - \frac{\rho\square t^3 - c}{2t(c-t)}\Bigg]\left[c_2\left\{1 - \left(\frac{c-t}{t}\right)^{\frac{1}{2}}\sin^{-1}\left(\frac{t}{c}\right)^{\frac{1}{2}} + \left(\frac{c-t}{t}\right)^{\frac{1}{2}}\frac{c_1}{\square}\right\}\right]^2, \tag{35}$$

$$\sigma_{22} = \frac{1}{3}t^2\left(\frac{c-t)}{t}\right)^{\frac{1}{2}} \square \frac{\rho \square t^3 - c}{2t(c-t)} - \frac{1}{t}\Bigg], \tag{36}$$

$$\sigma_{33} = \sin^2\theta\sigma_{22}. \tag{37}$$

For the second solution, the expansion factor is

$$\Theta = \frac{2(1-\alpha^2t^2)^{\frac{1}{2}}}{\alpha t^2}. \tag{38}$$

The components of the shear-tensor are given by

$$\sigma_{11} = \frac{2(1-\alpha^2t^2)^{\frac{1}{2}}}{3\alpha t^2}, \tag{39}$$

$$\sigma_{22} = -\frac{(1-\alpha^2t^2)^{\frac{1}{2}}}{3\alpha}, \tag{40}$$

$$\sigma_{33} = \sin^2\theta\sigma_{22}. \tag{41}$$

For the third solution, we have

$$\Theta = (y)^{\frac{1}{2}}\left(\frac{\dot{z}}{2z} + \frac{2}{t}\right). \tag{42}$$

The components of the shear-tensor are given by

$$\sigma_{11} = \frac{2}{3}\left(\frac{1}{t} - \frac{\dot{z}}{2z}\right)z\sqrt{y}, \tag{43}$$

$$\sigma_{22} = \frac{1}{3}t^2\left(\frac{\dot{z}}{2z} - \frac{1}{t}\right)\sqrt{y}, \tag{44}$$

$$\sigma_{33} = \sin^2\theta\sigma_{22}. \tag{45}$$

The shear invariant turns out to be 3 for all the solutions.

The rate of change of expansion with respect to proper time is given by Raychaudhuri's equation [12]

$$\frac{d\Theta}{d\tau} = -\frac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}u^au^b - R_{ab}u^au^b. \tag{46}$$

We evaluate it only for the second solution as it is simple to understand. For this solution it becomes

$$\frac{d\Theta}{d\tau} = -\frac{2}{\alpha^2}\left(\frac{1+\alpha^2t^2}{t^4}\right). \tag{47}$$

V. Summary

This section presents a brief summary of the results obtained in the previous section. We shall discuss these results and will comment on possible future directions of development.

We have been able to solve partially the EFEs for the class of non-static spherically symmetric spacetimes using different equations of state for the perfect fluid. These have been classified into three categories. The summary of each case is given below:

1. $\mathbf{R} = \mathbf{R}(\mathbf{t}),\ \lambda = \lambda(\mathbf{t})$

In this case, the dust solution gives

$$ds^2 = \frac{t}{c-t}dt^2 - \left[c_2\left\{1 - \left(\frac{c-t}{t}\right)^{\frac{1}{2}}\sin^{-1}\left(\frac{t}{c}\right)^{\frac{1}{2}} + \left(\frac{c-t}{t}\right)^{\frac{1}{2}}\frac{c_1}{\square}\right\}\right]^2 dr^2 - t^2 d\ ^2. \quad (48)$$

The stiff matter solution becomes

$$ds^2 = \frac{\alpha^2 t^2}{1-\alpha^2 t^2}dt^2 - dr^2 - t^2 d\ ^2. \quad (49)$$

2. $\mathbf{R} = \mathbf{R}(\mathbf{t}),\ \lambda = \lambda(\mathbf{r},\mathbf{t})$

The stiff matter solution gives

$$ds^2 = [y(t)]^{-1}dt^2 - z^2(r,t)dr^2 - t^2 d\ ^2, \quad (50)$$

where y and z are already given.

3. $\mathbf{R} = \mathbf{R}(\mathbf{r},\mathbf{t})$

In this case we obtain two solutions. The first solution, in fact, turns out a class of spherically symmetric static spacetimes. The other (stiff matter) solution is exactly the same as Eq. (49).

The non-static spherically symmetric solutions with equations of state have been split up into three classes of solutions. In the first case we obtain two different solutions. One of these two becomes the dust solution and the other becomes a stiff matter solution. The pressure and the energy density are positive every where. In case two we have a stiff matter solution only. However, it is very difficult to comprehend it. In the 3rd case we obtain two solutions. The first solution becomes a class of spherically symmetric static spacetimes depending upon the arbitrary function f. If we choose $f = \frac{1}{\sqrt{1-r^2}}$, it reduces to the Einstein spacetime. The energy density is positive everywhere while the pressure is negative. The other (stiff matter) coincides with one of the solutions in the first case. It is interesting to note that all of the non-static solutions which have been found contain shear; there are a very few solutions with shear in the literature.

Finally, we discuss the behaviour of the rate of change of expansion for the one solution given by Eq. (49). We see from Eq. (47) that the rate will never be positive; it will always be negative. As the time t tends to zero, the rate approaches to $-\infty$ and as t goes to ∞, the rate tends to zero. This shows that the spacetime is contracting or collapsing and the flux gets focussed along the proper time. The other solutions can be discussed similarly.

We have tried to obtain non-static spherically symmetric solutions for some particular classes and partial solutions have been attempted in these three classes. Thus a total of three solutions have been obtained. To obtain more new solutions one has to solve the remaining cases of these classes. Then we can attempt to find the general solution of the non-static spherically symmetric solutions.

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