Energy of the Gravitational Field of Slowly Rotating Neutron Stars

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Using the Einstein-Tolman expression for the energy-momentum pseudo-tensor, the energy density of the gravitational field of slowly rotating neutron stars was calculated in Cartesian coordinates. The energy density of the gravitational field could be expressed by the metric of the non-rotating neutron star and the angular velocity of the local inertial frames, accurate to the first order in the angular velocity of the star. A numerical method for calculating the energy density is given.

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I. INTRODUCTION

Rotating relativistic stars, such as neutron stars, are of fundamental interest in high-energy astrophysics [1-3]. The study of such stars not only can yield information about the equation of state (EOS) of matter at extremely high densities, but also provides a laboratory for the study of gravitational radiation. At present many studies are focused on the bulk properties of rotating neutron stars, such as the masses, radii, moments of inertia, and redshifts [1,4]. Another important topic is the study of the strong gravitational field of rotating neutron stars. This study is also very important for the research of the bulk properties and the composition of the neutron star. In this paper, we will use the Einstein-Tolman expression for the energy-momentum pseudo-tensor to calculate the energy density of the gravitational field of rotating neutron stars in Cartesian coordinates. The purpose of this study is to try to understand the energy of the gravitational field of rotating neutron stars.

As we know, if the energy density is calculated in spherical coordinates, the total energy of the static spherically symmetric stars will be infinite [5]. So one believes that spherical coordinates are not the proper ones to use. The energy and energy flux of cylindrical gravitational waves are positive and finite only in Cartesian coordinates [6,7], similarly the energy density of the gravitational field of a spherical distribution of matter also ought to be calculated in Cartesian coordinates.
II. THE EXPRESSION FOR THE ENERGY OF THE GRAVITATIONAL FIELD AND THE ENERGY DENSITY OF THE GRAVITATIONAL FIELD OF A NON-ROTATING NEUTRON STAR

According to the Einstein-Tolman expression, the energy-momentum pseudo-tensor of the gravitational field can be written as [6]

\[ \theta^\nu_\mu = \frac{1}{16\pi} H^\nu_{\mu\sigma} - (-g)^{1/2} T^\nu_\mu, \]

where

\[ H^\nu_{\mu\sigma} = (-g)^{-1/2} g_{\mu\lambda} \left[ g(\gamma^\nu_{\lambda\gamma} - g^\nu_{\sigma\gamma}) \right], \gamma, \]

and \( T^\nu_\mu \) is the energy-momentum tensor of the perfect fluid, expressed as

\[ T^\nu_\mu = (p + \rho) u^\nu u_\mu - p \delta^\nu_\mu, \]

in which \( u^\nu \) are four-velocities, here and throughout we set \( G = c = 1 \).

The metric of a non-rotating neutron star has the form

\[ ds^2 = e^{2\nu} dt^2 - e^{2\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

The energy density of the gravitational field of the non-rotating neutron star in Cartesian coordinates, which has been calculated in our previous work [8], is

\[ \theta^0_0 = (re^\nu)^{1/2} (p + \rho) \frac{M(r)}{[r - 2M(r)]^{3/2}}. \]

It is evident that the energy density of the gravitational field is positive and rational. In order to get the total energy of the system, integrating the total energy density over the whole space, \( P_0 = \int \frac{1}{16\pi} H^\nu_{0\sigma} dV \), one gets the relation \( P_0 = M \), which means that the total inertial mass of the system just equals gravitation mass, where \( P_0 \) is the total energy of the system, denoting the inertial mass; \( M \) is the source of gravitation, denoting the gravitation mass.

III. HARTLE’S APPROXIMATION OF A SLOWLY ROTATING NEUTRON STAR

In relativity, the space-time geometry of a rotating star in equilibrium is described by a stationary axisymmetric metric of the form

\[ ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - e^{\mu} d\theta^2 - e^{\phi} (d\phi + \omega dt)^2, \]

where \( \nu, \lambda, \mu, \phi, \) and \( \omega \) are function of the coordinates \( r \) and \( \theta \) only.

When the angular velocity \( \Omega \) is far smaller than the Kepler frequency \( \Omega_k \), accurate to order \( \Omega \), Eq. (6) becomes

\[ ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + 2\omega r^2 \sin^2 \theta d\phi dt, \]
where \( \nu, \lambda, \) and \( \omega \) are function of \( r \) only.

Retaining only the first-order terms in \( \Omega \), one gets a differential equation for the angular velocity of the fluid relative to the local inertial frame as [9]

\[
\frac{1}{r^4} \frac{d}{dr} \left( r^4 \frac{d\varpi}{dr} \right) + \frac{4}{r} \frac{dj}{dr} \varpi = 0,
\]

(8)

in which \( \varpi = \Omega - \omega \), \( j(r) = e^{-(\nu+\lambda)/2} \).

Using a multipole expansion of the metric functions, Hartle et al. gave the corrections of the metric (7) up to quadrupole order as [9,10]

\[
ds^2 = e^\nu [1 + 2(h_0 + h_2 P_2)] dt^2 - e^\lambda \left[ 1 + \frac{2(m_0 + m_2 P_2)}{(r - 2M)} \right] dr^2 - r^2 [1 + 2(\nu_2 - h_2) P_2] [d\theta^2 + \sin^2 \theta (d\phi^2 - 2\omega \theta d\phi dt)],
\]

(9)

in which \( P_2 \) is the Legendre polynomial of order 2; \( M, e^\nu, \) and \( e^\lambda \) are the mass and metric functions of the non-rotating neutron star with the same central density; \( \nu \) is proportional to the star’s angular velocity \( \Omega \); \( h_0, h_2, m_0, m_2, \) and \( \nu_2 \) are functions of \( r \) and proportional to \( \Omega^2 \). Based on this approximation, they gave a method for getting numerical results for slowly rotating neutron stars in general relativity [10].

From Eq. (8), one can find that the angular velocity of the local inertial frames, which are dragged along in the direction of the star’s rotation, increases as the radius of the neutron star decreases [1], that is \( d\omega/dr < 0 \). For the stiffer equation of state (EOS), when \( \Omega \to \Omega_k \), the ratio \( \omega_k/\Omega \) could be up to 0.9 at the centre of the neutron star, which indicates that there is a strong dragging effect for the inertial frames [11]. Another effect of the rotation is the increase of the mass and the radius of the neutron star with the same central density, the increase could be up to about 20% for the mass and about 10% for the radius [3].

IV. ENERGY DENSITY OF THE GRAVITATIONAL FIELD OF A SLOWLY ROTATING NEUTRON STAR IN CARTESIAN COORDINATES

From Eq. (7), the components of the covariant metric tensor of a slowly rotating star in spherical coordinates can be written as

\[
g_{00} = e^\nu, \quad g_{11} = -e^\lambda, \quad g_{22} = -r^2 \sin^2 \theta, \quad g_{33} = -r^2, \quad g_{02} = \omega r^2 \sin^2 \theta.
\]

(10)

Retaining only first-order terms in \( \Omega \), the components of the contravariant tensor are

\[
g^{00} = e^{-\nu}, \quad g^{11} = -e^{-\lambda}, \quad g^{22} = -\frac{1}{r^2 \sin^2 \theta}, \quad g^{33} = -\frac{1}{r^2}, \quad g^{02} = e^{-\nu} \omega.
\]

(11)

In Eq. (7), replacing \( r, \theta, \) and \( \phi \) by the Cartesian coordinates \( x, y, \) and \( z, \) with \( x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = \cos \theta, \) we obtain the components of the covariant metric
tensor in Cartesian coordinates (accurate to one order in $\Omega$):

$$g_{00} = e^{-\nu},$$  \hspace{1cm} (12a)

$$g_{11} = -(e^\lambda \cos^2 \phi \sin^2 \theta + \sin^2 \phi + \cos^2 \phi \cos^2 \theta),$$  \hspace{1cm} (12b)

$$g_{22} = -(e^\lambda \sin^2 \phi \sin^2 \theta + \cos^2 \phi + \sin^2 \phi \cos^2 \theta),$$  \hspace{1cm} (12c)

$$g_{33} = -(e^\lambda \cos^2 \theta + \sin^2 \theta),$$  \hspace{1cm} (12d)

$$g_{12} = -\sin \phi \cos \phi \sin^2 \theta(e^\lambda - 1),$$  \hspace{1cm} (12e)

$$g_{13} = -\cos \phi \sin \theta \cos \theta(e^\lambda - 1),$$  \hspace{1cm} (12f)

$$g_{23} = -\sin \phi \sin \theta \cos \theta(e^\lambda - 1),$$  \hspace{1cm} (12g)

$$g_{01} = -\omega r \sin \phi \sin \theta,$$  \hspace{1cm} (12h)

$$g_{02} = -\omega r \cos \phi \sin \theta.$$  \hspace{1cm} (12i)

The determinant of the metric is $g \approx -e^{\nu+\lambda}$ (accurate to one order in $\Omega$). Using the coordinate transformation equations [7]

$$g^{\alpha\beta} = \frac{\partial x^\alpha}{\partial \xi^\mu} \frac{\partial x^\beta}{\partial \xi^\nu} g_{\mu\nu},$$

in which $\alpha, \beta, \mu, \nu = 0, 1, 2, 3$, $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $\xi^0 = t$, $\xi^1 = r$, $\xi^2 = \phi$, $\xi^3 = \theta$, $g_{\mu\nu}$ are the components of the contravariant tensor in spherical coordinates, we obtain the components of the contravariant tensor in Cartesian coordinates:

$$g^{00} = e^{-\nu},$$  \hspace{1cm} (13a)

$$g^{11} = -(e^\lambda \cos^2 \phi \sin^2 \theta + \sin^2 \phi + \cos^2 \phi \cos^2 \theta),$$  \hspace{1cm} (13b)

$$g^{22} = -(e^\lambda \sin^2 \phi \sin^2 \theta + \cos^2 \phi + \sin^2 \phi \cos^2 \theta),$$  \hspace{1cm} (13c)

$$g^{33} = -(e^\lambda \cos^2 \theta + \sin^2 \theta),$$  \hspace{1cm} (13d)

$$g^{12} = -\sin \phi \cos \phi \sin^2 \theta(e^{-\lambda} - 1),$$  \hspace{1cm} (13e)

$$g^{13} = -\cos \phi \sin \theta \cos \theta(e^{-\lambda} - 1),$$  \hspace{1cm} (13f)

$$g^{23} = -\sin \phi \sin \theta \cos \theta(e^{-\lambda} - 1),$$  \hspace{1cm} (13g)

$$g^{01} = -\omega r \sin \phi \sin \theta e^{-\nu},$$ \hspace{1cm} (13h)

$$g^{02} = -\omega r \cos \phi \sin \theta e^{-\nu}.$$  \hspace{1cm} (13i)

From Eqs. (2), (12), and (13), we obtain the components of $H_0^0$ in Cartesian coordinates (accurate to one order in $\Omega$):

$$H_0^{00} = 0,$$  \hspace{1cm} (14a)

$$H_0^{01} = \frac{2x}{r^2} e^{\frac{1}{2}(\nu-\lambda)}(e^\lambda - 1) - \frac{x}{r} \omega \rho e^{\frac{1}{2}(\nu+\lambda)},$$  \hspace{1cm} (14b)

$$H_0^{02} = \frac{2y}{r^2} e^{\frac{1}{2}(\nu-\lambda)}(e^\lambda - 1) - \frac{y}{r} \omega \rho e^{\frac{1}{2}(\nu+\lambda)},$$  \hspace{1cm} (14c)

$$H_0^{03} = \frac{2z}{r^2} e^{\frac{1}{2}(\nu-\lambda)}(e^\lambda - 1) - \frac{z}{r} \omega \rho e^{\frac{1}{2}(\nu+\lambda)}.$$  \hspace{1cm} (14d)
Here and throughout, a function followed by a prime (such as $\omega'$) denotes a partial derivative of the function with respect to the radius $r$.

From Eqs. (1), (2), and (14) we obtain the total energy density of the slowly rotating neutron star:

$$t_0^0 = \frac{1}{16\pi} H^0_{0,\sigma}$$

$$= \frac{1}{16\pi} \left\{ \frac{1}{r} e^{\frac{1}{2}(\nu - \lambda)} \left[ \frac{2}{r} (e^\lambda - 1) + (\nu' + \lambda')(e^\lambda - 1) + 2\lambda' \right] 
- \sin^2 \theta e^{\frac{1}{2}(\nu + \lambda)} r^2 \left[ \frac{3}{r} \omega' + \omega'^2 + \omega \omega'' - \frac{1}{2}(\nu' + \lambda')\omega' \right] \right\}. \quad (15)$$

The energy density of the gravitational field can be written as

$$\theta_0^0 = t_0^0 - (-g)^{1/2} T_0^0,$$ \quad (16)

where $T_0^0 = (P + \rho) u_0^0 - P \delta_0^0$. Since $u_0^0 = e^{-\nu/2}[1 + \frac{1}{2}r^2 \sin^2 \theta \omega^2 e^{-\nu} - h_0 - h_2 P_2]$ [10], accurate to one order in $\Omega$, one has $T_0^0 \approx \rho$. From Eq. (9) we know that $h_0, h_2, m_0, m_2,$ and $\nu_2$ are proportional to $\Omega^2$, so, accurate to one order in $\Omega$, the metric functions $e^\nu, e^\lambda$ of a slowly rotating neutron star can be replaced by those of the metric of the non-rotating neutron star. For non-rotating stars one has [12]

$$\lambda' = \frac{1}{r} \left( 1 - e^\lambda + 8\pi r^2 e^\lambda \right), \quad (17)$$

$$\nu' = \frac{1}{r} \left( e^\lambda - 1 + 8\pi r^2 e^\lambda \right), \quad (18)$$

$$e^{-\lambda} = 1 - \frac{2M(r)}{r}, \quad (19)$$

where $M(r) = \int_0^r 4\pi r^2 \rho dr$. Substituting Eqs. (17)–(19) into Eq. (15), and substituting the result into Eq. (16), we obtain the energy density of the gravitational field of the slowly rotating neutron star as

$$\theta_0^0 = (re^\nu)^{1/2}(p + \rho) \frac{M(r)}{[r - 2M(r)]^{3/2}}$$

$$- \frac{1}{16\pi} \sin^2 \theta e^{-\frac{1}{2}(\nu + \lambda)} r^2 \left[ \frac{3}{r} \omega' + \omega'^2 + \omega \omega'' - 4\pi r(p + \rho)\omega' e^\lambda \right]. \quad (20)$$

Comparing this result to Eq. (4), one can see that the difference of the energy density of the gravitational field between the rotating neutron star and non-rotating neutron star is the second part of Eq. (20), that is

$$\delta\theta_0^0 = \theta_0^0(\text{non}) - \theta_0^0(\text{rot})$$

$$= \frac{1}{16\pi} \sin^2 \theta e^{-\frac{1}{2}(\nu + \lambda)} r^2 \left[ \frac{3}{r} \omega' + \omega'^2 + \omega \omega'' - 4\pi r(p + \rho)\omega' e^\lambda \right]. \quad (21)$$
As the expression of the energy density of the gravitational field of rotating neutron stars
is accurate to one order in $\Omega$, while the density, pressure, mass, and metric in Eq. (20) are
function of $r$ and $\theta$ in powers of $\Omega^2$ in Hartle’s method [9], and thus accurate to one order
in $\Omega$, the first part of Eq. (20) could be numerical calculated as in non-rotating neutron
stars. According to the OV equation [13] and the equation of state, given a central density
of matter, one can get the numerical result for the density, pressure, mass, and metric as
functions of $r$, and then get the numerical result for the first part of Eq. (20). As to the
second part of Eq. (20), one can solve Eq. (8) to get the numerical result for $\omega$, $\omega'$, and
$\omega''$ as function of $r$ [10] (there are two free parameters: the central density and the central
angular velocity relative to the local inertial frame). In this way, one can get the numerical
result for the second part of Eq. (20). As a prediction, we estimate that the energy density
of the gravitational field of the rotating neutron star is very large and could be comparable
to the energy density of matter, that is nearly $10^{14}$ g cm$^{-3}$.

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