Localized Modes and the Vibrations of Elastic Structures

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We consider the theory of the localized vibrational modes that can exist at the ends or edges of elastic structures. We show that for certain geometries and special values of Poisson's ratio, these modes can be perfectly localized, and not radiate into the structure. This localization has interesting effects on the way that the vibrational patterns and frequencies of the normal modes of a structure are changed when the dimensions of the structure are altered.

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I. INTRODUCTION

In this article, we give a general discussion of some interesting properties of the vibrational modes of elastic structures. Recently, there has been a number of investigations of these modes using ultrafast optical techniques [1–4]. In a typical experiment, a pump light pulse is directed at a nanostructure sample. When the light is absorbed, a stress is suddenly set up, and the structure begins to vibrate. A probe light pulse at a time delay t is used to detect these vibrations, usually through a measurement of the dependence of the reflected intensity ΔR on t. A detailed theory of the dependence of ΔR on t is very difficult to construct. This is because the amplitude and phase of the different modes depends in a complicated way on the details of how the pump light is absorbed, for example, in which parts of the structure the absorption is strongest. In addition, the signal detected by the probe light is affected by how the probe interacts with the sample and hence it is dependent on the optical properties of the sample which may not be accurately known. Despite these difficulties, from an analysis of the Fourier transform of $\Delta R(t)$, the frequencies of the vibrational modes can be found directly.

If the elastic properties of the sample are known, one can then try to use the measured frequencies of the normal modes to determine the geometry of the nanostructure. This becomes an interesting mathematical problem of the general type discussed by M. Kac [5] in a famous paper "Can you hear the sound of a drum?" Kac considered an elastic membrane that was under a uniform tension. If the frequencies of all the normal modes are known, can one deduce the shape of the boundary of the membrane? The answer is no, and explicit examples of membranes that have different shapes but have the same mode frequencies have been given [6, 7].

At the more practical level, one can try to proceed by assuming that the sample is reasonably approximated by some simple geometrical shape, such as a cylinder, and then determine the parameters of the shape by adjusting these parameters so that the best fit to the normal mode frequencies is obtained. It is surprisingly difficult to do this, and as we will discuss in this paper, this is because in many cases, the frequencies have a complex dependence on the shape parameters. This complexity arises because of the existence of modes that are localized at the ends or corners of the structures. This is discussed in detail with several examples in a recently submitted paper [8]; here we give a condensed description of the principal results and some further examples and discussions.

II. MODES OF A ROD

We consider the vibrations of structures composed of elastically isotropic materials. To find the normal modes, we use either the basis function method [9] or commercial software ABAQUS [10]. For the particular geometries that we report here, these methods are able to give reliable results for the mode patterns and frequencies.

As a first example, we show in Fig. 1 the frequencies of the normal modes of a cylindrical rod of length L and radius R, and with Poisson's ratio of $\sigma = 0.25$. The frequencies shown are for the modes in which the radial displacement is independent of the azimuthal angle and the strain associated with the mode has even parity along the length of the rod.

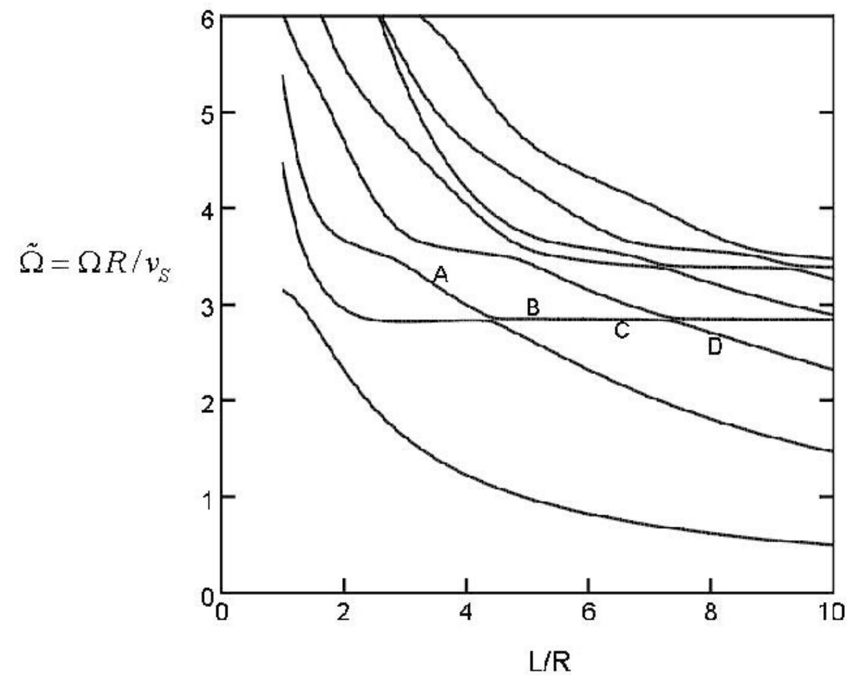

FIG. 1: Dimensionless frequency $\tilde{\Omega} = \Omega R/v_S$ of the normal modes of a rod as a function of the ratio of the rod length L to the radius R. The transverse sound velocity is v_S and Poisson's ratio is taken to be 0.25. Only the eight lowest frequency modes are shown.

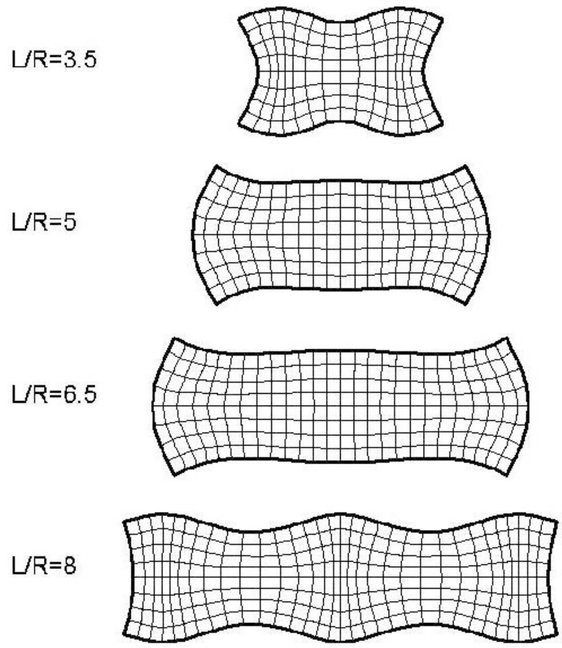

FIG. 2: Vibration patterns of the third normal mode shown in Fig. 1 for different values of the ratio of the rod length L to the radius R.

The dimensionless frequency $\tilde{\Omega} \equiv \Omega R/v_s$ is plotted, where v_s is the shear wave velocity. A striking feature of the result is the existence of the frequency plateaus, the first being at $\tilde{\Omega} = 2.848$. In Fig. 2, we show the mode patterns for the vibrations at the points A, B, C and D where $L/R = 3.5$, 5, 6.5 and 8, respectively. One can see that for A and D, the vibration extends along the whole length of the rod, whereas for B and C, the vibration is localized to near the ends. For B and C, the displacement is large at the ends, then decreases approximately exponentially. A more detailed examination of the mode pattern (this is not so easy to see from the figure) reveals that in the center section of the rod the displacement varies periodically with distance z and has an almost constant amplitude. This suggests the following interpretation. We suppose that it is possible to find a solution of the equations of elasticity that is localized near to the end and approximately satisfies the boundary conditions at the free surfaces of the rod. Since this is not

$$\tilde{\Omega} = \Omega\, R/v_S v_S$$

an exact solution, we call it a quasi-localized mode QLM. Because this does not exactly satisfy the boundary conditions, if the mode is excited, it will lose amplitude as a result of radiation down the length of the rod. This acoustic radiation will excite the QLM at the other end of the rod.

In Fig. 3, we show the dispersion relation for waves propagating along a cylindrical rod. These are calculated from the Pochhammer equations [11]. Only modes of azimuthal symmetry are shown. One can see immediately from this plot that for a QLM at the

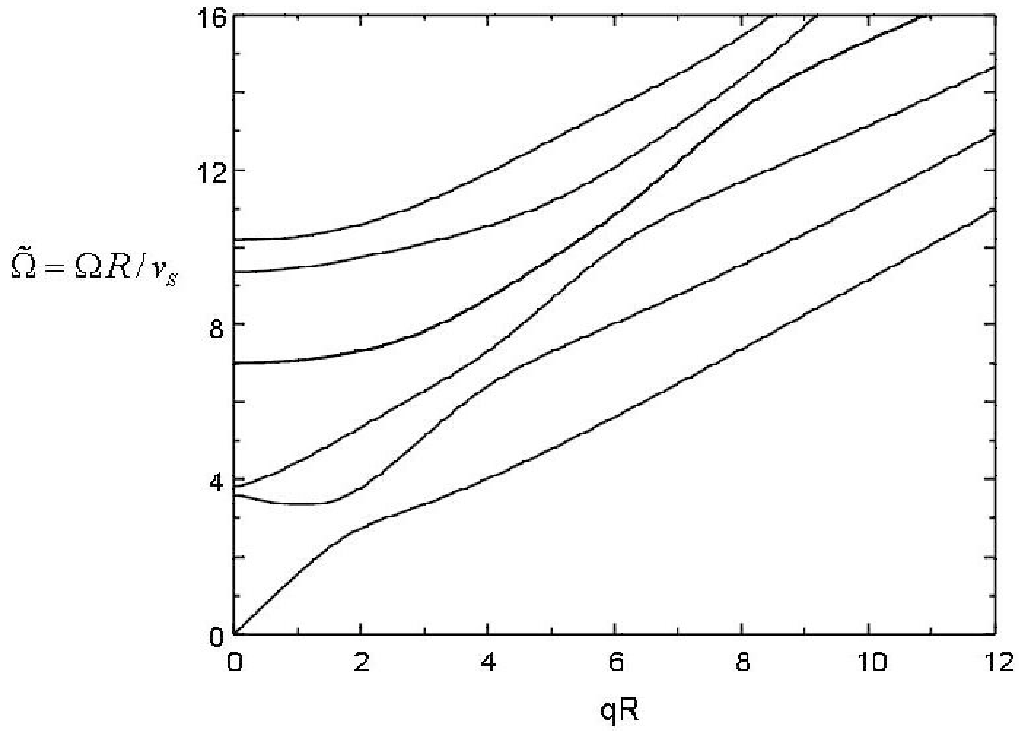

FIG. 3: Dimensionless frequency $\tilde{\Omega} = \Omega R/v_S$ for waves of wave number q propagating along a rod with circular cross-section of radius R. The transverse sound velocity is v_S and Poisson's ratio is 0.25.

frequency of the first plateau, there can be coupling to only the lowest frequency propagating mode. The wave number of this mode is found to be $q = 2.151/R$, and so the phase velocity of this wave is $v_p = 1.324v_s$ and the wavelength is $\lambda = 2.92R$. Then it follows that if the length L of the rod is increased by one wavelength, i.e., by $\Delta L = 2.92R$, then the frequency and mode pattern will be unchanged. This is in agreement with the length of the plateaus as shown in Fig. 1.

In Ref. 8, we have considered the vibrations of bars and plates for which similar results are obtained.

One can make a simple mechanical model to capture the essential physics, and this is shown in Fig. 4. We consider two masses M connected by springs of strength K to an elastic rod of length L. The springs are attached to points that are a distance C from each end of the rod. One can show that the frequencies of the normal modes of this system are the solution of the equation

$$(K - M\Omega^2)\left[K\sin(\alpha) + \frac{Eq\cos(qL/2)}{\cos(qC)}\right] = K^2\sin(\alpha),$$

$$\tilde{\Omega} = \Omega\, R/v_S v_S$$

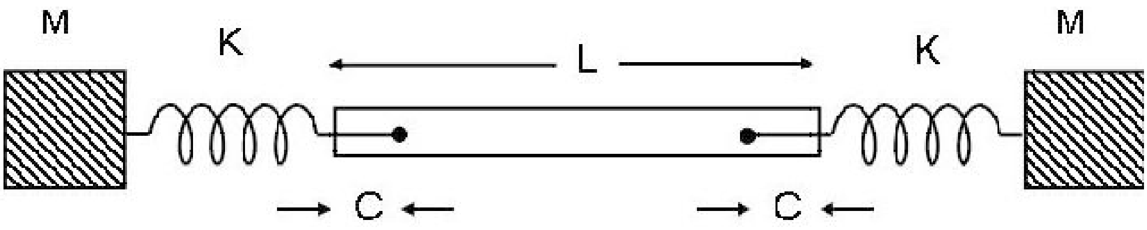

FIG. 4: Simplified model for the coupling of localized modes to propagating waves in a bar.

where $\alpha = q(L - 2C)/2$. By a suitable choice of the parameters E, K, M, and C, one can obtain an excellent fit to the numerical results shown in Fig. 1, and to the results obtained for bars and plates.

III. MODES THAT ARE PERFECTLY LOCALIZED

As discussed above, the quasi-localized modes lose energy by coupling to waves that can propagate along the rod. Consequently, if a QLM at the end of a semi-infinite rod is excited, the amplitude of vibration at the end of the rod will decrease with time as $\exp(-\Gamma t)$, where Γ measures the coupling between the QLM and the propagating mode. In the specific example discussed above, there is only one propagating mode to which the QLM can couple. If Poisson's ratio σ is varied, the amplitude of this coupling will vary, and for a critical value, σ_c may become zero. While it is clear that the coupling will vary, one cannot see *a priori* that the coupling will actually become zero for any value of Poisson's ratio in the allowable range. However, it turns out that in fact for a rod, the coupling does become zero at $\sigma_c = 0.158$. When this coupling becomes zero the localized mode will not radiate any energy away from the end of the rod; in this situation we will call it a perfectly localized mode (PLM). For such a mode, the vibration pattern is localized near the end of the rod and varies periodically with time without any damping.

It is then interesting to consider whether it is possible to modify the shape of the end of a rod in a way such that there is a perfectly localized mode for other values of σ. Of course, there are many different shapes for the end of a rod that one could consider. Numerical results for one particular family of shapes are shown in Fig. 5. The change in the end of the rod consists of the addition of a disk of thickness c and radius $c + R$ to the end of the rod. By varying the ratio of c to R, one can find a PLM for any chosen value of σ in the range below 0.158. We have explored some other shapes, and it appears likely that by considering a sufficiently wide class of variations in shape, it should be possible to find a PLM for any value of σ up to the maximum possible value of 0.5. For a description of results obtained for PLM at the ends of bars and plates, see Ref. 8.

In Ref. 8, we report some examples of shape variations in which it is possible to find a PLM over a certain range of values of σ but not outside this range. A detailed investigation revealed that this happened because when the shape was varied to obtain a PLM, the

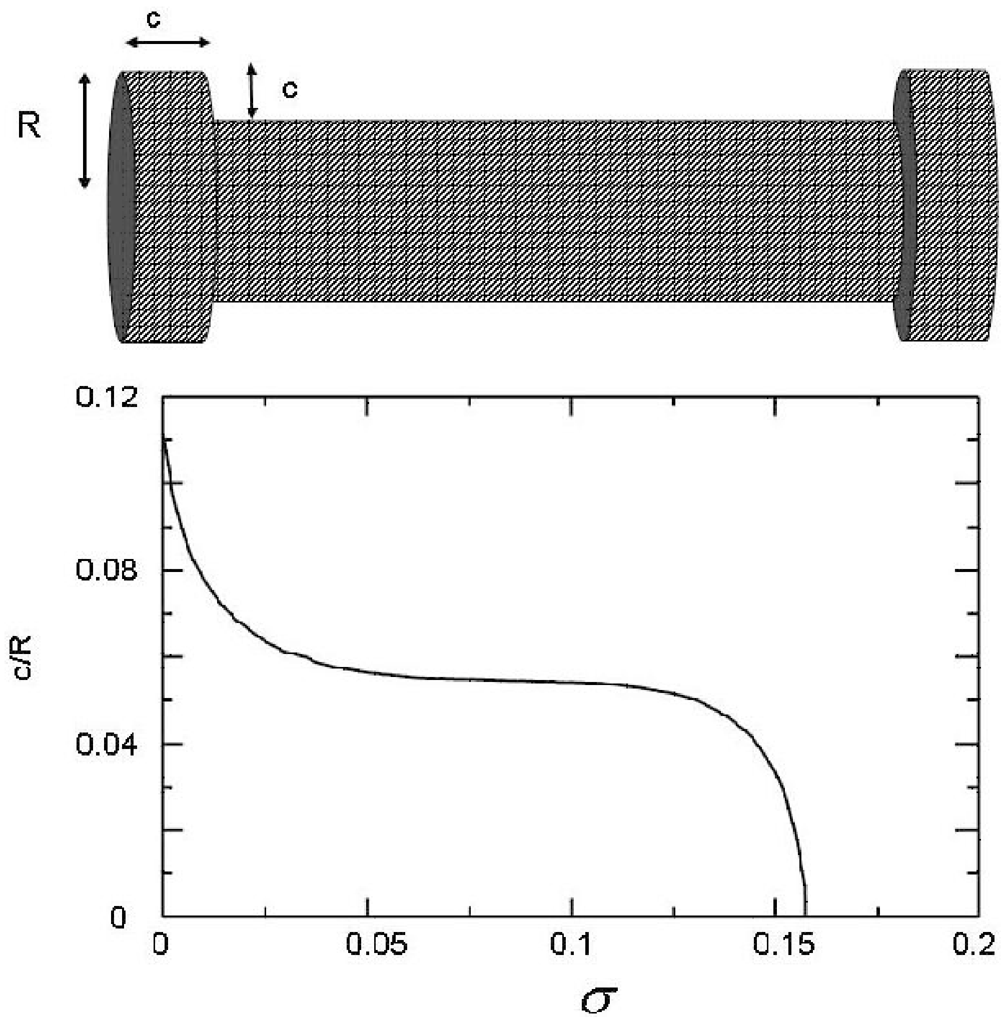

FIG. 5: The effect of varying the shape of the ends of the rod. The lower plot shows the ratio c/R needed to produce a perfectly localized mode as a function of Poisson's ratio σ.

frequency of the PLM might increase to a value such that the localized mode could couple to two propagating modes. Then in order to prevent the localized mode from radiating, the coupling to both of these propagating modes had to vanish. Naturally, this could not be achieved through the variation of a single shape parameter, but presumably by varying two parameters, it should be possible. For example, in the shape shown in Fig. 5, it might be possible to find a PLM by varying independently the thickness of the disk and the amount by which its radius exceeds the radius of the rod.

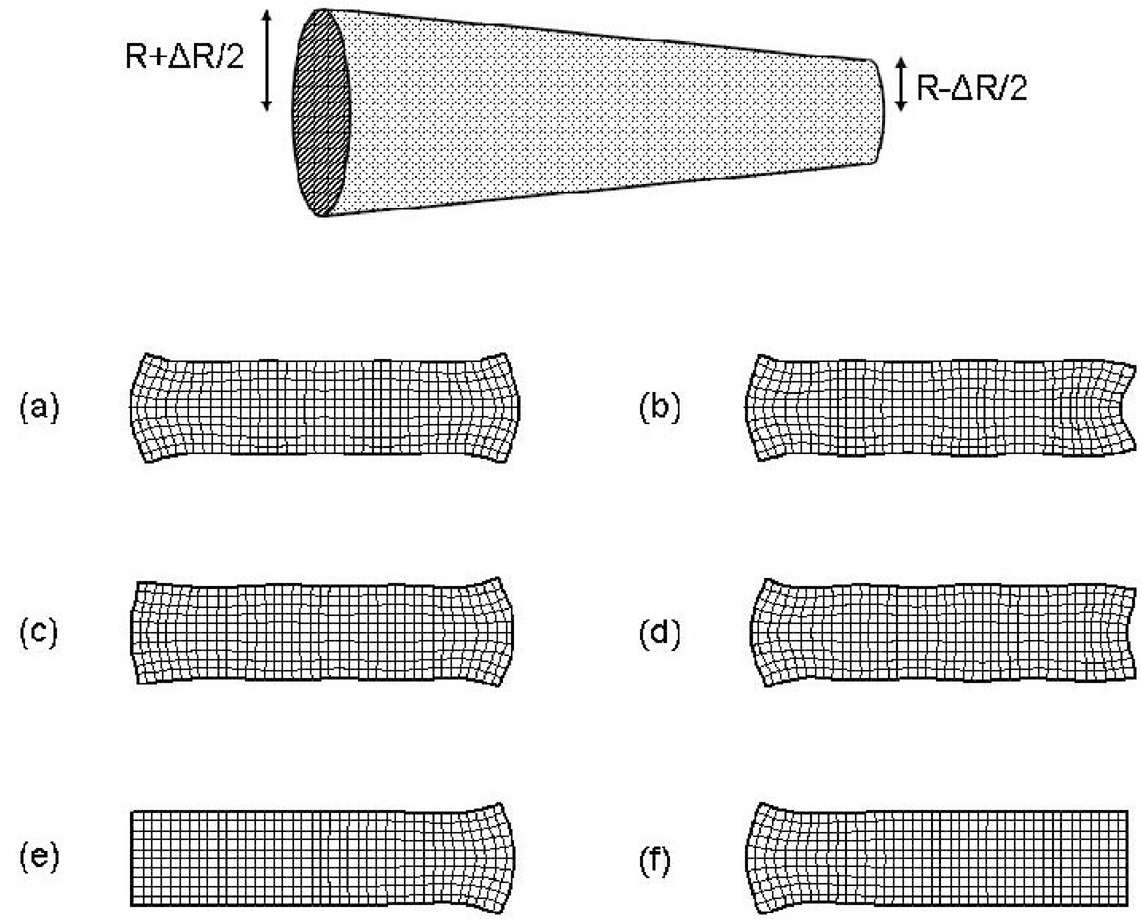

FIG. 6: a) and b) show the mode patterns for modes symmetric and anti-symmetric along the length, respectively, when the radii at the two ends are equal. Poissons ratio is 0.35. The patterns of these modes when $\Delta R/R = 0.01$ are shown in c) and d) for $\sigma = 0.35$, and in e) and f) for $\sigma = 0.14$.

IV. LOCALIZATION AND THE VARIATION OF MODE FREQUENCIES WITH SHAPE AND SIZE

The existence of QLM has an important effect on the way the mode frequencies change when the size or shape of a structure is altered. At first sight, it would appear that the effect would simply be that since the QLM is confined to some region of the structure, changes in the geometry outside this region will have no effect on the frequency or vibration pattern of the QLM. This can provide an explanation of the plateau regions shown in Fig. 1. However, based on this view, a plateau would extend indefinitely, and there would not be the breaks between the successive plateaus. One can understand the existence of these breaks with reference to the model of Fig. 3. For certain values of the length of the rod a wave that is emitted by the QLM at the left hand end of the rod, travels to the right down the rod, is reflected, and then returns with a phase such that it adds constructively to the next wave sent out by the left hand QLM. As a result, a standing wave of the rod is resonantly excited, giving an increased coupling between the QLM and the waves that can propagate along the rod. An alternative, and equivalent, view is to consider that the breaks in the plateau are the result of "hybridization" between the QLM modes and the extended modes in the rod.

The gaps between successive plateaus are larger if the QLM is more strongly coupled to the propagating mode; if the QLM was perfectly localized, there would be no gaps for a long rod. As a specific example of the effect of localization, consider the modes of a rod that has a radius varying along its length from $R + \Delta R/2$ to $R - \Delta R/2$ as shown in Fig. 6. In this example,

$$\Delta R/R = 0.01\sigma = 0.35\sigma = 0.14$$

the length of the rod is taken to be eight times the radius. If both the ends of the rod have the same radius ($\Delta R = 0$), then the vibrational amplitudes at each end will be the same and there will be modes that have even and odd parity along the rod (Figs. 6a and 6b). In Figs. 6c and 6d, we show how these modes change when $\Delta R/R$ equals 0.01 when $\sigma = 0.35$. For this value of σ, the QLM are strongly coupled to the propagating mode, and so even though the QLM at opposite ends of the rod have slightly different frequencies, they are still coupled together through the propagating mode. Consequently, the modes of the structure have a vibration pattern that has close to even or odd parity. For $\sigma = 0.14$, on the other hand (Figs. 6e and 6f), the coupling to the propagating mode is much weaker and a difference between the end radii of $0.01R$ results in the mode pattern changing so that the vibration is localized almost completely at one end of the bar or at the other.

V. SUMMARY

We have given a discussion of the normal modes of nanostructures and how the vibration patterns and frequencies of these modes change when the size and shape of the structure is modified. We show that the presence of modes that are localized near to the ends or corners of structures has a large effect on these changes. We have demonstrated that for special values of Poisson's ratio or for special shapes of the ends of a rod, perfectly localized modes can exist, i.e., modes that do not lose any energy due to radiation into the bulk of the structure.

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References

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[1] G. A. Antonelli, H. J. Maris, S. G. Malhotra and J. M. E. Harper, J. Appl. Phys. **91**, 3261 (2002).

[2] R. Taubert, F. Hudert, A. Bartels, F. Merkt, A. Habenichtt, P. Leiderer, and T. Dekorsy, New J. Phys. **9**, 376 (2007).

[3] C. Giannetti, B. Revaz, F. Banfi, M. Montagnese, G. Ferrini, F. Cilento, S. Maccalli, P. Vavassori, G. Oliviero, E. Bontempi, L. E. Depero, V. Metlushko, and F. Parmigiani, Phys. Rev. B **76**, 125413 (2007).

[4] T. Bienville, J. F. Robillard, L. Belliard, I. Roch-Jeune, A. Devos, and B. Perrin, Ultrasonics **44**, Suppl. 1, E1289 (2006).

[5] M. Kac, Am. Math. Monthly **73**, 1 (1966).

[6] J. Milnor, Proc. Nat. Acad. Sci. **51**, 542 (1964).

[7] S. Zelditch, Geom. Funct. Anal. **10**, 628 (2000).

[8] J. Ma and H. J. Maris, submitted for publication.

[9] W. M. Visscher, A. Migliori, T. M. Bell, and R. A. Reinert, J. Acoust. Soc. Am. **90**, 2154 (1991).

[10] ABAQUS software from Dassault Systèmes.

[11] L. Pochhammer, Z. Reine Ang. Math. **81**, 324 (1876).