Statistical Properties of the Excited Binomial States for the Pseudoharmonic Oscillator

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In this paper we have studied the binomial states and also the excited binomial states defined in terms of the Fock-basis vectors of the pseudoharmonic oscillator. We have demonstrated that outside of the behavior at the harmonic limit, when the binomial states lead to the coherent states of the one dimensional harmonic oscillator, these states have in fact all the characteristics of the coherent states defined on the complex unit disk. We have calculated the expectation values and the Mandel parameter (as well as their thermal analogue) which give us information on their statistical behavior. All the obtained relations tend, at the harmonic limit, to the corresponding relations for the one dimensional harmonic oscillator.

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I. INTRODUCTION

The coherent states (CSs), $|z;\lambda\rangle$, like the number states $|n;\lambda\rangle$ play an important role in quantum optics. (See, e.g., [1] and the references therein.)

Generally a CS has the following structure:

$$|z,\lambda\rangle = \frac{1}{\sqrt{N\left(|z|^2;\lambda\right)}}\sum_{n=0}^{M}\frac{z^n}{\sqrt{\rho(n;\lambda)}}|n,\lambda\rangle, \tag{1}$$

where $z = |z|\exp(i\varphi)$ is a complex variable labeling the CS, $\rho(n;\lambda)$ are positive quantities depending on the energy eigenvalues and are called the structure constants, λ is a parameter which characterizes the CSs (e.g., the Bargmann index, the quantum rotational number J, and so on) and M is a positive integer, $M \leq \infty$.

There are many families of CSs depending of the different choice of structure constants $\rho(n;\lambda)$. The only condition is that the normalized function $N^{(M)}(|z|;\lambda)$ must be finite:

$$N^{(M)}\left(|z|^2;\lambda\right) \equiv \sum_{n=0}^{M}\frac{(|z|^2)^n}{\rho(n;\lambda)} < \infty. \tag{2}$$

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The binomial states (BSs), which were introduced by Stoler *et al.* [1], are defined as a linear superposition of finite number states (NSs) $|n;\lambda\rangle$ in an $(M+1)$-dimensional subspace of the infinite dimensional Hilbert space of the Fock-vectors:

$$|z, M; \lambda; 0\rangle = \sum_{n=0}^{M} B_n^{(M;0)} z^n (1-|z|^2)^{\frac{M-n}{2}} |n;\lambda\rangle, \quad B_n^{(M;0)} \equiv \sqrt{\begin{pmatrix} M \\ n \end{pmatrix}}. \tag{3}$$

Here the complex variable is defined in the unit disk $D = \{z \in C, z < 1\}$. If z becomes a real number, then it plays the role of probability. With the orthogonality relation for the Fock-vector's basis $\langle n;\lambda|n';\lambda\rangle = \delta_{nn'}$, and using Newton's binomial relation, we can normalize the BSs to unity:

$$\langle z, M; \lambda; 0|z, M; \lambda; 0\rangle = \sum_{n=0}^{M} \begin{pmatrix} M \\ n \end{pmatrix} \left(|z|^2\right)^n (1-|z|^2)^{M-n} = 1. \tag{4}$$

The name "binomial states" comes from the fact that the corresponding photon distribution is simply the binomial distribution with the probability $|z|^2$:

$$P_n^{(M;0)}(|z|^2) \equiv |\langle n;\lambda|z, M;\lambda;0\rangle|^2 = \begin{pmatrix} M \\ n \end{pmatrix} \left(|z|^2\right)^n (1-|z|^2)^{M-n}. \tag{5}$$

As we can see, the Fock-basis space for BSs covers only a subspace of (M+1)-dimension of the whole infinite-dimensional Fock-space. In addition, the BSs are independent of the Fock-vector basis $|n;\lambda\rangle$ we use, i.e., the weighting functions $C_n^{(M;0)}(z) \equiv \langle n;\lambda|z, M;\lambda;0\rangle = B_n^{(M;0)} z^n \sqrt{(1-|z|^2)^{M-n}}$ are independent of the basis choice. But, when we will calculate the expectation values for a concrete physical system, then the choice of the basis becomes important.

From the above, it can be observed that the BSs are only a particular case of CSs. In this context, we must show that BSs satisfy all the properties required for the CSs.

The aim of our paper is to examine the properties of the binomial and excited binomial states built in the Fock-vector basis $\{|n;\lambda\rangle, n = 0, 1, \ldots, M\}$ of the pseudoharmonic oscillator (PHO). The paper is organized as follows. In Sec. II we give some main elements and concepts related to the PHO that will be needed in the next sections. In Sec. III we build the binomial states for the PHO by repeatedly applying the raising operator K_+ on the fiducial (or reference) BS $|z, M; k; 0\rangle$, and we will demonstrate that these states are indeed coherent states, satisfying all the prescriptions needed for the coherent states. In Sec. IV we introduce, build, and examine some properties of the excited binomial states (EBSs) for the PHO, while Sec. V is devoted to the examination of the nonclassical behavior of these states. In Sec. VI we investigate some statistical properties of the superposition of EBSs for the PHO with thermal light, while in the Sec. VII we examine the non-excited limit of these states (which corresponds to $m = 0$) and also the harmonic limit of EBSs for the PHO, which leads to the corresponding characteristics of the one dimensional harmonic oscillator (HO-1D).

II. PSEUDOHARMONIC OSCILLATOR AND THEIR ASSOCIATED QUANTUM GROUP

In order to construct the BSs for the pseudoharmonic oscillator (PHO), let us review some elements related to this oscillator which will be useful in the next sections. A suitable physical system which can be correctly modeled by the PHO is the diatomic molecule, e.g., the effective potential of the PHO is [2, 3]

$$V_J(r) = \frac{m_{\rm red}\omega^2}{8} r_0^2 \left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2 + \frac{\hbar^2}{2m_{\rm red}} J(J+1)\frac{1}{r^2}, \tag{6}$$

where $m_{\rm red}$ and ω are, respectively, the reduced mass and the angular frequency of the PHO, r_0 is the equilibrium distance, and $J = 0, 1, 2, \ldots$ is the rotational quantum number.

We consider the Hilbert space as a functional realization of the Fock space with basis $\{|n;k\rangle,\ n = 0,1,2,\ldots\}$. The number $2k = \pm1, \pm2, \pm3, \ldots$ is called the Bargmann index, as we will see later. The PHO potential is of the central field kind, so the functions $\Psi_{nJm}(\vec{r}) = \langle\vec{r}|n;k\rangle = R_{nJ}(r)Y_{Jm}(\theta,\varphi)$ give a realization of this basis in the "coordinate" representation, where m_z is the quantum number of the operator J_z.

The Fock-basis vectors fulfill the orthogonality and completeness relations:

$$\langle n;k|n';k\rangle = \delta_{nn'}, \quad \sum_{n=0}^{\infty} |n;k\rangle\langle n;k| = 1. \tag{7}$$

The resolution of the Schrödinger equation for the stationary states $H\Psi_{nJm}(\vec{r}) = E_{nJ}\Psi_{nJm}(\vec{r})$ leads to following energy spectrum for the PHO, which is linear with respect to the vibrational quantum number n:

$$E_{nJ} = \hbar\omega\,(n+k) - \frac{m_{\rm red}\omega^2}{4} r_0^2 \equiv E_{0J} + \hbar\omega n. \tag{8}$$

The rotational quantum number J is "embedded" in the Bargmann index $k = k(J)$, i.e., [3]

$$k = \frac{1}{2} + \frac{1}{2}\sqrt{\left(J+\frac{1}{2}\right)^2 + \left(\frac{m_{\rm red}\omega}{2\hbar} r_0^2\right)^2}. \tag{9}$$

The radial eigenfunctions corresponding to the stationary states are [3]

$$\begin{aligned} u_{nJ}(r) &\equiv rR_{nJ}(r) \\ &= \left[\frac{m_r\omega}{\hbar}\frac{n!}{2^{2k-1}\Gamma(n+2k)}\right]^{\frac{1}{2}} \left(\frac{m_r\omega}{\hbar} r\right)^{2k-\frac{1}{2}} \\ &\quad \exp\left[-\frac{1}{4}\left(\frac{m_r\omega}{\hbar}\right)^2 r^2\right] L_n^{2k-1}\left[\frac{1}{2}\left(\frac{m_r\omega}{\hbar}\right)^2 r^2\right], \end{aligned} \tag{10}$$

where $\Gamma(x)$ is Euler's gamma function and $L_n^{2k-1}(x)$ is the generalized Laguerre polynomial.

The natural dynamical group associated with the bound vibrational states of the PHO is SU(1,1) [3], whose discrete representations are given by

$$K^2|n;k\rangle = k(k-1)|n;k\rangle, \tag{11}$$

$$K_+|n;k\rangle = \sqrt{(n+1)(n+2k)}|n+1;k\rangle, \tag{12}$$

$$K_-|n;k\rangle = \sqrt{n(n+2k-1)}|n-1;k\rangle, \tag{13}$$

where K, K_+, and K_- are the Casimir, the raising, and, respectively, the lowering operators. Here the real parameter λ from Sec. I is just the Bargmann index k. Even if the Bargmann index k can take negative values, here we are interested only in the positive series of discrete representations of SU(1,1).

The commutation relations of the group generators K_+, K_-, and K_3 look like

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_3. \tag{14}$$

We define also the self-adjoint operator N "numbering" basic elements (also called the particle (photon or boson) number operator):

$$N = K_3 - k, \quad N|n;k\rangle = n|n;k\rangle. \tag{15}$$

In the next section we will concentrate our attention on the BSs of the PHO labeled by points $z \in D$, defined as

$$|z, M; k; 0\rangle = \sum_{n=0}^{M} B_n^{(M;0)} z^n \left(1 - |z|^2\right)^{\frac{M-n}{2}} |n;k\rangle, \tag{16}$$

i.e., on the states constructed by using $M+1$ Fock-vectors $\{|n;\lambda\rangle, n = 0, 1, \ldots, M\}$ for the PHO.

III. BINOMIAL STATES FOR THE PHO

Based on the concrete oscillator case (the PHO), we will show that, even if at the harmonic limit any BS tends to the CS of the HO-1D, the BSs behave as coherent states even up to this limit. For this purpose, in order to highlight this statement, we must prove that the BSs of the PHO satisfies the minimal Klauder prescriptions imposed on the CSs (see, e.g., [4, 5]), which means that a BS must be: a continuous function in the complex z variable, i.e., the map $z \in C \to |z, M; k; 0\rangle \in L^2(R)$ must be continuous; be normalized but not orthogonal; fulfill the resolution of the identity operator with a positive weight function; be temporally stable and fulfill the action identity.

The basic minimum properties of any set of states to be called a set of coherent states (which were formulated by Klauder [4, 5]) are the following:

Proposition 3.1: If $|z, M; k; 0\rangle$ is a set of BSs for the PHO, then the continuity of labeling requires that the norm of the difference of two BSs, i.e. $\| |z', M; k; 0\rangle - |z, M; k; 0\rangle \| \to 0$ whenever $z' \to z$.

Proof 3.1: Explicitly, the norm of the difference of two BSs is

$$\left\| |z', M; k; 0\rangle - |z, M; k; 0\rangle \right\| = \sqrt{2} \left[1 - \mathrm{Re} \left(\langle z, M; k; 0 | z', M; k; 0\rangle \right)\right]^{\frac{1}{2}} . \tag{17}$$

The overlap relation between two BSs is given through the scalar product:

$$\langle z, M; k; 0 | z', M; k; 0\rangle = \sum_{n=0}^{M} \left[B_n^{(M;0)} \right]^2 \left[\left(1 - |z|^2\right) \left(1 - |z'|^2\right) \right]^{\frac{M-n}{2}} \left(z^* z'\right)^n, \tag{18}$$

from which we can see that if $z' \to z$, then the overlap tends to unity and the continuity is fulfilled.

Proposition 3.2: The BSs are normalized but not orthogonal.

Proof 3.2: This condition is implicitly shown from the above overlap relation, and so the non-orthogonallity is proved.

Proposition 3.3: The BSs fulfilled the resolution of the identity operator with the positive weight function

$$\int d\mu_0^{(M)}(z; k) |z, M; k; 0\rangle \langle z, M; k; 0| = I_{M+1}. \tag{19}$$

Proof 3.3: Let us assume that there exists a positive integration measure $d\mu_0^{(M)}(z; k)$ on $L^2(R)$ of the following form:

$$d\mu_0^{(M)}(z; k) = \frac{d^2 z}{\pi} h_0^{(M)}(|z|^2; k) = \frac{d\varphi}{2\pi} d(|z|^2) h_0^{(M)}(|z|^2; k), \tag{20}$$

where $\frac{d^2 z}{\pi}$ is the Lebesgue measure on D, and the weight function $h_0^{(M)}(|z|^2; k)$ will be determined as a positive function for $|z| < 1$.

After the angular integration, which leads to

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} (z^*)^n z^{n'} = (|z|^2)^n \delta_{nn'}, \tag{21}$$

and using the completeness relation of the Fock vectors in an $(M + 1)$ - dimensional space, i.e.,

$$\sum_{n=0}^{M} |n; k\rangle \langle n; k| = I_{M+1}, \tag{22}$$

we have to solve the following integral equation for the unknown weight function $h_0^{(M)}(|z|^2; k)$:

$$\int_0^1 d(|z|^2) h_0^{(M)}(|z|^2; k) (|z|^2)^n (1 - |z|^2)^{M-n} = \frac{1}{\begin{pmatrix} M \\ n \end{pmatrix}} = \frac{\Gamma(n+1)\Gamma(M+1-n)}{\Gamma(M+1)}. \tag{23}$$

If we perform the variable change, $|z|^2 (1-|z|^2)^{-1} \equiv X$, we are lead to the equation

$$\int_0^\infty dX \frac{h_0^{(M)}(X;k)}{(1+X)^{M+2}} X^n = \frac{\Gamma(n+1)\Gamma(M+1-n)}{\Gamma(M+1)}. \tag{24}$$

This requires a function change,

$$g_0^{(M)}(X;k) = \Gamma(M+1)\frac{h_0^{(M)}(X;k)}{(1+X)^{M+2}}, \tag{25}$$

and also the index change $n = s-1$, in order to lead to the Stieltjes moment problem [5]:

$$\int_0^\infty dX h_0^{(M)}(X;k) X^{s-1} = \Gamma(s)\Gamma(M+2-s). \tag{26}$$

The solution of such a problem can be expressed through Meijer's G-functions [6, 7]:

$$g_0^{(M)}(X;k) = G_{11}^{11}\left(X \left| \begin{array}{c} -M-1 \\ 0 \end{array} \right.\right) = \Gamma(M+2)\frac{1}{(1+X)^{M+2}}, \tag{27}$$

so that the integration measure for the BSs becomes

$$d\mu_0^{(M)}(z;k) = (M+1)\frac{d\varphi}{2\pi} d(|z|^2), \tag{28}$$

and evidently, their weight function is positive. The proof is finished.

Proposition 3.4: The BSs are temporally stable, which means that any BS always remains a BS, during the time evolution:

$$e^{-\frac{\mathrm{i}}{\hbar}Ht}|z,M;k;0\rangle = |z(t),M;k;0\rangle. \tag{29}$$

Proof 3.4: By using the time independent Schrödinger equation for the PHO,

$$H|n;k\rangle = E_{nJ}|n;k\rangle, \tag{30}$$

and the expression of the energy spectrum (which is linear with respect to the vibrational quantum number n), we obtain, successively,

$$e^{-\frac{\mathrm{i}}{\hbar}Ht}|z,M;k;0\rangle = e^{-\frac{\mathrm{i}}{\hbar}E_{0J}t}\sum_{n=0}^{M} B_n^{(M;0)}(ze^{-\mathrm{i}\omega t})^n \left(1-\left|ze^{-\mathrm{i}\omega t}\right|^2\right)^{\frac{M-n}{2}} |n;k\rangle \equiv |z(t),M;k;0\rangle, \tag{31}$$

with $z(t) = |z|\, e^{-\mathrm{i}(\varphi+\omega t)} ze^{-\mathrm{i}\omega t}$.

Proposition 3.5: The BSs fulfill the action identity

$$\langle z,M;k;0|H|z,M;k;0\rangle \approx \omega|\alpha|^2. \tag{32}$$

Proof 3.5: If in the expectation (mean) value of the Hamiltonian in the BSs representation we apply Newton's binomial formula, finally we obtain

$$\langle z,M;k;0|H|z,M;k;0\rangle = \sum_{n=0}^{M} \left(\begin{array}{c} M \\ n \end{array} \right) \left(|z|^2\right)^n \left(1-|z|^2\right)^{M-n} E_{nJ} = E_{0J} + \hbar\omega M\,|z|^2\,. \tag{33}$$

Now, if we use the notation $M\,|z|^2 = |\alpha|^2$ (by considering $\hbar = 1$), then we obtain, up to an additional constant related with the rotational energy, the so-called action identity [5, 8].

This result establishes that the expectation value (or "lower symbol") of the Hamiltonian mimics the classical relation energy-action, i.e., the real variable $|\alpha|^2 = M|z|^2$ can be identified with an action variable, canonical with the angular variable $\omega = \dot{\varphi}$ [5].

IV. EXCITED BINOMIAL STATES FOR THE PHO

As in the cases referring to the canonical coherent states (CCSs) for the HO-1D, we introduce the excited binomial states (EBSs) by repeatedly (m fold, where m is a positive integer) application of the SU(1,1) raising operator on the non-excited ("fiducial") BS $|z,M;k;0\rangle$, in perfect analogy with the definition of the photon added CSs, see [9] (for the HO - 1D) or [10] (for the PHO):

$$|z,M;k;m\rangle \equiv N_m^{(M)}(|z|^2)\,(K_+)^m\,|z,M;k;0\rangle. \tag{34}$$

The action of the raising operator on the basis states is

$$(K_+)^m|n;k\rangle = \sqrt{\frac{\Gamma(n+m+1)\Gamma(n+2k+m)}{\Gamma(n+1)\Gamma(n+2k)}}|n+m;k\rangle. \tag{35}$$

If we introduce the short notation

$$B_n^{(M;m)} \equiv \sqrt{\Gamma(M+1)\frac{\Gamma(n+m+1)\Gamma(n+2k+m)}{\Gamma(M+1-n)\left[\Gamma(n+1)\right]^2\Gamma(n+2k)}}, \tag{36}$$

then the EBSs become

$$|z,M;k;m\rangle \equiv N_m^{(M)}(|z|^2)\sum_{n=0}^{M} B_n^{(M;m)} z^n \left(1-|z|^2\right)^{\frac{M-n}{2}} |n+m;k\rangle. \tag{37}$$

The normalization function $N_m^{(M)}(|z|^2)$ (with the evident condition $N_0^{(M)}(|z|^2) = 1$) is obtained if we impose that the EBSs must be normalized to unity: $\langle z,M;k;m|z,M;k;m\rangle = 1$.

In order to deduce the normalization function, it is useful to examine the following sum:

$$S_0^{(M;m)}(X) \equiv \sum_{n=0}^{M} \left[B_n^{(M;m)}\right]^2 X^n, \tag{38}$$

implicitly, using the properties of the shifted factorials (or Pochhammer symbols), as well as the binomial coefficients [6, 7]:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)}. \tag{39}$$

So we obtain successively, expressing the sum through the hypergeometric ${}_pF_q(\ldots;X)$ and Meijer G-functions $G_{pq}^{mn}(X|\ldots)$

$$\begin{aligned} S_0^{(M;m)}(X) &= \frac{\Gamma(2k+m)\Gamma(m+1)}{\Gamma(2k)} {}_3F_2(-M, m+1, 2k+m; 1, 2k; -X) \\ &= \frac{1}{\Gamma(-M)} G_{33}^{13}\left(X \,\middle|\, \begin{matrix} M+1, & -m, & 1-2k-m; \\ 0; & 1-2k, & 0 \end{matrix}\right). \end{aligned} \tag{40}$$

Finally, the normalization function is

$$\begin{aligned} N_m^{(M)}(|z|^2) &= \frac{1}{(1-|z|^2)^{\frac{M}{2}}} \frac{1}{\sqrt{S_0^{(M;m)}(X)}} \\ &= \frac{\sqrt{\Gamma(-M)}}{(1-|z|^2)^{\frac{M}{2}}} \frac{1}{\sqrt{G_{33}^{13}\left(X \,\middle|\, \begin{matrix} M+1, & -m, & 1-2k-m; \\ 0; & 1-2k, & 0 \end{matrix}\right)}}, \end{aligned} \tag{41}$$

and also the EBSs for the PHO are

$$|z, M; k; m\rangle \equiv \frac{1}{\sqrt{S_0^{(M;m)}(X)}} \sum_{n=0}^{M} B_n^{(M;m)} \left(\frac{z}{\sqrt{1-|z|^2}}\right)^n |n+m; k\rangle. \tag{42}$$

Let us now perform the resolution of the identity operator.

$$\int d\mu_m^{(M)}(z;k) |z, M; k; m\rangle\langle z, M; k; m| = I_{M+1}, \tag{43}$$

where the integration measure with the unknown weight function $h_m^{(M)}(|z|^2;k)$ is

$$d\mu_m^{(M)}(z;k) = \frac{d^2z}{\pi} h_m^{(M)}(|z|^2;k) = \frac{d\varphi}{2\pi} d(|z|^2) h_m^{(M)}(|z|^2;k). \tag{44}$$

After suitable function changes (similar to those in the case of BSs for the PHO, Sec. III) and using the properties of Meijer's G-functions [6, 7], the integration measure of the EBSs becomes

$$d\mu_m^{(M)}(z;k) = \frac{1}{\Gamma(M+1)} \frac{d\varphi}{2\pi} \frac{d(|z|^2)}{(1-|z|^2)^2} S_0^{(M;m)}(X) G_{33}^{31}\left(\frac{|z|^2}{1-|z|^2} \middle| \begin{array}{ccc} M-1; & m, & 2k-1+m \\ 0, & 0, & 2k-1; \end{array} \right). \tag{45}$$

Moreover it is not difficult to demonstrate that this integration measure ensures the validity of the identity operator resolution.

At the end of this section let us we calculate the photon (boson) probability distribution, i.e., the probability of finding $n+m$ particles (photons, bosons) in the EBS $|z, M; \lambda; m\rangle$:

$$P_{n+m}^{(M;m)}(|z|^2;k) = |\langle n+m;\lambda|z,M;\lambda;m\rangle|^2 = \frac{1}{S_0^{(M;m)}(X)} \left[B_{n-m}^{(M;m)}\right]^2 X^{n-m}. \tag{46}$$

V. NON-CLASSICAL BEHAVIOR OF THE EBSs

Generally, the expectation (or mean) value of a physical observable which characterize a quantum system connected with a PHO, in the ECSs representation, can be expressed as

$$\begin{aligned} \langle z, M; k; m|A|z, M; k; m\rangle &\equiv \langle A\rangle_z^{(M;m)} \\ &= \left[N_m^{(M)}(|z|^2)\right]^2 \sum_{n,n'=0}^{M} B_n^{(M;m)} B_{n'}^{(M;m)} z^n (z^*)^{n'} \\ &\quad \left(1-|z|^2\right)^{M-\frac{n+n'}{2}} \langle n'+m;k|A|n+m;k\rangle. \end{aligned} \tag{47}$$

We will provide particular attention to the expectations of integer powers ($s = 1, 2, \ldots$) of the particle number operator N:

$$N^s|n+m;k\rangle = (n+m)^s|n+m;k\rangle. \tag{48}$$

The expectation value becomes

$$\langle N^s\rangle_z^{(M;m)} = \frac{1}{S_0^{(m)}(X)} \sum_{l=0}^{s} \binom{s}{l} m^{s-l} S_l^{(M;m)}(X), \tag{49}$$

where we have used the short notation

$$\begin{aligned} S_l^{(M;m)}(X) &\equiv \sum_{n=0}^{M} B_n^{(M;m)} X^n n^l = \left(X\frac{d}{dX}\right)^l S_0^{(M;m)}(X) \\ &= \frac{1}{\Gamma(-M)} \left(X\frac{d}{dX}\right)^l G_{33}^{13}\left(X \middle| \begin{array}{ccc} M+1, & -m, & 1-2k-m; \\ 0; & 1-2k, & 0 \end{array} \right). \end{aligned} \tag{50}$$

In order to use the differential properties of Meijer's G-functions [6], we must pay attention to the following differentiation equality:

$$\left(X\frac{d}{X}\right)^{l}(\ldots)=\sum_{j=0}^{l}c_j^{(l)}X^j\left(\frac{d}{X}\right)^{j}(\ldots)\,. \tag{51}$$

Here, the real coefficients $c_j^{(l)}$ are the same as those from the development of the integer power of a natural number n^s, written as a finite series [10]:

$$n^l=\sum_{j=0}^{l}c_j^{(l)}\binom{n}{j}j!\,. \tag{52}$$

If we use the short notation (with $j=0,1,2,\ldots$):

$$G_{33}^{13}\left(X\,\middle|\,\begin{matrix}M+1, & -m, & 1-2k-m;\\ 0; & 1-2k, & 0\end{matrix}\right)=G_{33}^{13}\left(X\,\middle|\,\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right), \tag{53}$$

then for our case we obtain

$$\begin{aligned}\left(X\frac{d}{X}\right)^{l}G_{33}^{13}\left(X\,\middle|\,\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right) &= \sum_{j=0}^{l}c_j^{(l)}X^j\left(\frac{d}{X}\right)^{j}G_{33}^{13}\left(X\,\middle|\,\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right)\\ &= \sum_{j=0}^{l}c_j^{(l)}G_{44}^{14}\left(X\,\middle|\,\begin{matrix}0, & M+1, & -m, & 1-2k-m;\\ 0; & 1-2k, & 0, & j\end{matrix}\right)\\ &= \sum_{j=0}^{l}c_j^{(l)}G_{33}^{13}\left(X\,\middle|\,\begin{matrix}\ldots,(m)\\ \ldots,j\end{matrix}\right).\end{aligned} \tag{54}$$

Finally, the expectation value is

$$\langle N^s\rangle_z^{(M;m)}=\frac{\sum\limits_{l=0}^{s}\binom{s}{l}m^{s-l}\left[\sum\limits_{j=0}^{l}c_j^{(l)}G_{33}^{13}\left(X\,\middle|\,\begin{matrix}M+1, & -m, & 1-2k-m;\\ 0; & 1-2k, & j\end{matrix}\right)\right]}{G_{33}^{13}\left(X\,\middle|\,\begin{matrix}M+1, & -m, & 1-2k-m;\\ 0; & 1-2k, & 0\end{matrix}\right)}, \tag{55}$$

or, in short notation

$$\langle N^s\rangle_z^{(M;m)}=\frac{\sum\limits_{l=0}^{s}\binom{s}{l}m^{s-l}\left[\sum\limits_{j=0}^{l}c_j^{(l)}G_{33}^{13}\left(X\,\middle|\,\begin{matrix}\ldots,(m)\\ \ldots,j\end{matrix}\right)\right]}{G_{33}^{13}\left(X\,\middle|\,\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right)}. \tag{56}$$

A useful way to measure the deviation from the Poisson distribution of the number of particles and to characterize the corresponding quantum field is via the Mandel $Q_z^{(M;m)}$ parameter, defined as [8, 11]

$$Q_z^{(M;m)} \equiv \frac{V_z^{(M;m)}}{\langle N \rangle_z^{(M;m)}} - 1 = \frac{\langle N^2 \rangle_z^{(M;m)} - \left(\langle N \rangle_z^{(M;m)} \right)^2}{\langle N \rangle_z^{(M;m)}} - 1, \tag{57}$$

where $V_z^{(M;m)}$ is the variance of the particle (photon or boson) distribution, which measures the deviation from the Poisson distribution. This is always a positive quantity: $V_z^{(M;m)} \equiv \langle (N - \langle N \rangle)^2 \rangle_z^{(M;m)} \geq 0$.

The eigenvalue n of the number-particle operator N is regarded as the number of photons (bosons) or intensity emitted within a certain time interval by a light source.

Depending on the values of the Mandel parameter, as a function of the variable $|z|^2$, it can compute the areas where this function is negative, zero, or positive, so the field has a sub-Poissonian, Poissonian, respectively super-Poissonian behaviour.

Propositon 5.1: The Mandel parameter $Q_z^{(M;m)}$ *compares the fluctuations of the particle number operator to that of a Poissonian source. If* $Q_z^{(M;m)} < 0, = 0,$ *respectively* > 0, *the field is called sub-Poissonian, Poissonian, respectively super-Poissonian.*

Proof 5.1: In order to demonstrate this assertion, we appeal to the CSs of a one-dimensional harmonic oscillator (HO-1D):

$$|z\rangle = \exp\left(-\frac{|z|^2}{2}\right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{58}$$

with number states $|n\rangle \equiv |n; \lambda = 0\rangle$ in our notation of Sec. I.

A Poissonian source is an ideal laser which generates the coherent states of the HO-1D.

The mean value of integer power of the particle-number operator N^s in a CS of the HO-1D is

$$\langle N^s \rangle = e^{-|z|^2} \left(|z|^2 \frac{d}{d|z|^2} \right)^s e^{|z|^2}, \tag{59}$$

so that the Mandel parameter for a Poissonian source (ideal laser) is

$$Q_z = \frac{\langle N^2 \rangle_z - (\rangle N \rangle_z)^2}{\langle N \rangle_z} - 1 = 0. \tag{60}$$

The corresponding weighting function is

$$P_n(|z|^2) \equiv |\langle n|z \rangle|^2 = \exp\left(-|z|^2\right) \frac{\left(|z|^2\right)^n}{n!} = \exp\left(-\langle N \rangle\right) \frac{(\langle N \rangle)^n}{n!}, \tag{61}$$

i.e., the Poissonian distribution in the variable $\langle N \rangle = |z|^2$.

Consequently, the coherent states have Poissonian behavior, and the Mandel parameter vanishes for the Poisson distribution.

Generally, the sub-Poissonian field has the fluctuations or photon count noise smaller than that of coherent (ideal laser) light with the same intensity. The most sub-Poissonian states are the number-states $|n\rangle$ of particles with boson statistics, for which $\langle N^s\rangle_n \equiv \langle n|N^s|n\rangle = n^s$, so that the corresponding Mandel parameter is $Q_n = -1$, i.e., the greatest possible negative value allowed for the Mandel parameter, because the variance is always positive.

Ultimately, as an example of a super-Poissonian state, is the chaotic state, whereas the field which has photon-count noise higher than the coherent-light noise is called a super-Poissonian field. Thermal radiation is often called chaotic light because it has no order, as we shall see later.

For the EBSs of the PHO, the Mandel $Q_z^{(M;m)}$ parameter becomes

$$Q_z^{(M;m)} = \frac{G_{33}^{13}\left(X\left|\begin{matrix}\ldots,(m)\\ \ldots,2\end{matrix}\right.\right)G_{33}^{13}\left(X\left|\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right.\right)-\left[G_{33}^{13}\left(X\left|\begin{matrix}\ldots,(m)\\ \ldots,1\end{matrix}\right.\right)\right]^2-m\left[G_{33}^{13}\left(X\left|\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right.\right)\right]^2}{G_{33}^{13}\left(X\left|\begin{matrix}\ldots,(m)\\ \ldots,1\end{matrix}\right.\right)G_{33}^{13}\left(X\left|\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right.\right)+mG_{33}^{13}\left(X\left|\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right.\right)}. \quad (62)$$

Depending on the range of the complex variable $|z| < 1$ and also of the order of excitation m, it can compute the areas where the Mandel parameter is negative, zero, or positive, corresponding to the sub-Poissonian, Poissonian, or super-Poissonian behavior of the corresponding fields.

Using the differentiation properties of the Meijer functions and Eq. (40), it is useful to rewrite the above expression in the following manner:

$$Q_z^{(M;m)} = \frac{X^2\frac{d}{dX}\left[\frac{d}{dX}\ln G_{33}^{13}\left(X\left|\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right.\right)\right]-m}{X\left[\frac{d}{dX}\ln G_{33}^{13}\left(X\left|\begin{matrix}\ldots,(m)\\ \ldots,0\end{matrix}\right.\right)\right]+m} = \frac{X^2\frac{d}{dX}\left[\frac{d}{dX}\ln S_0^{(M;m)}(X)\right]-m}{X\left[\frac{d}{dX}\ln S_0^{(M;m)}(X)\right]+m}. \quad (63)$$

Because it is difficult to represent this general expression of the Mandel parameter as a function of $|z|$, in order to evince the behavior of the EBSs we adopt an approximate expression, obtained by expanding $\ln S_0^{(M;m)}(X)$ in a power series of the variable $X = |z|^2/(1-|z|^2)$ around $X = 0$ (i.e., for small $|z|^2$) and truncating the development at the first power of variable X. Using Eqs. (38) and (36), we obtain

$$\ln S_0^{(M;m)}(X) = \ln\left[B_0^{(M;m)}\right]^2 + X\frac{\left[B_1^{(M;m)}\right]^2}{\left[B_0^{(M;m)}\right]^2} + \ldots = \ln\Gamma(m+1) + M(m+1)\left(1+\frac{m}{2k}\right)X + \ldots . \quad (64)$$

So, in the approximation for small $|z|^2$, the Mandel parameter becomes

$$Q_z^{(M;m)} = \frac{-m}{M(m+1)\left(1+\frac{m}{2k}\right)\frac{|z|^2}{1-|z|^2}+m} < 0, \tag{65}$$

which shows that for small values of $|z|^2$, the behavior of BSs is sub-Poissonian.

Particularly, for $m = 0$ we can express the above Meijer functions through the Jacobi polynomials (see Appendix A):

$$\begin{aligned} & G_{33}^{13}\left(X \,\middle|\, \begin{matrix} M+1, & 0, & 1-2k; \\ 0; & 1-2k, & j \end{matrix}\right) \\ = \; & G_{22}^{12}\left(X \,\middle|\, \begin{matrix} M+1, & 0; \\ 0; & j \end{matrix}\right) \\ = \; & \Gamma(-M)\Gamma(j+1)(X+1)^M(-1)^j P_j^{(-M-1;-j)}\left(\frac{X-1}{X+1}\right) \\ = \; & (-1)^j\Gamma(-M+j)X^j(X+1)^{M-j}, \end{aligned} \tag{66}$$

and we can calculate exactly the Mandel parameter for the non-excited BSs:

$$Q_z^{(M;0)} = \frac{G_{22}^{12}\left(X \,\middle|\, \begin{matrix} M+1, & 0; \\ 0; & 2 \end{matrix}\right)}{G_{22}^{12}\left(X \,\middle|\, \begin{matrix} M+1, & 0; \\ 0; & 1 \end{matrix}\right)} - \frac{G_{22}^{12}\left(X \,\middle|\, \begin{matrix} M+1, & 0; \\ 0; & 1 \end{matrix}\right)}{G_{22}^{12}\left(X \,\middle|\, \begin{matrix} M+1, & 0; \\ 0; & 0 \end{matrix}\right)} = -|z|^2 < 0. \tag{67}$$

This result shows that the non-excited BSs have a sub-Poissonian behavior, and this statement is consistent with the results of other authors [1, 18].

VI. THERMAL STATES

Let us consider a thermal state of a PHO quantum system in thermodynamical equilibrium with the environment described by the global density operator

$$\rho = \frac{1}{Z(\beta)} \sum_{J=0}^{\infty} (2J+1) Z_J(\beta) \rho_J(\beta). \tag{68}$$

The reduced density operator which corresponds to a rotational state (a state with a fixed rotational quantum number$J$) is characterized by the canonical distribution function

$$\rho_J = \frac{1}{Z_J(\beta)} \sum_{n=0}^{\infty} e^{-\beta E_{nJ}} |n;k\rangle\langle n;k|, \tag{69}$$

with the partial (rotational) partition function

$$Z_J(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_{nJ}} = e^{-\beta E_{0J}} \sum_{n=0}^{\infty} \left(e^{-\beta\hbar\omega}\right)^n = e^{+\beta\frac{m_{red}\omega^2}{4} r_0^2}\,(\bar{n}+1)\left(\frac{\bar{n}}{\bar{n}+1}\right)^k, \tag{70}$$

where we have used the particle number mean value (Bose-Einstein distribution function):

$$\bar{n} = \frac{1}{e^{\beta\hbar\omega} - 1}, \quad \beta = (k_B T)^{-1}. \tag{71}$$

This approach shows that the "rotational" density operator ρ_J is, in fact, a reduced density operator of the whole density operator ρ, because we have summed over the vibrational quantum number n.

Moreover, the global (or whole) partition function is

$$Z(\beta) = \sum_{n,J=0}^{\infty} e^{-\beta E_{nJ}} = \sum_{J=0}^{\infty} (2J+1) Z_J(\beta), \tag{72}$$

where we have taken into account the rotational degeneration with the rotational degree of degeneration $g_J = 2J + 1$.

In order to reveal the properties of Husimi's Q- and P- functions attached to the PHO, we pay particular attention to the their rotational or reduced density operator ρ_J [12]:

$$\rho_J = \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n |n;k\rangle\langle n;k|. \tag{73}$$

i.e., we examine the "rotational" part of these distribution functions. This approach is justified because the EBSs $|z, M; k; m\rangle$ are dependent on the rotational quantum number through the Bargmann index $k = k(J)$.

In order to provide insight into the classical features of the radiation field, as well as for performing a statistical description of the quantum systems connected with a PHO, we will direct our attention to two important functions: the Husimi's Q- distribution function and also the Glauber – Sudarshan P- (quasi-)distribution function.

VI-1. Husimi's Q- distribution function

The "rotational" Husimi Q- distribution function attached to the reduced density operator ρ_J is defined through the expectation value

$$Q_J^{(M;m)}(|z|^2) \equiv \langle z, M; k; m|\rho_J|z, M; k; m\rangle. \tag{74}$$

Proposition 6.1: The Husimi Q- distribution function attached to the reduced density operator ρ_J of the PHO has the following expression:

$$\begin{aligned} Q_J^{(M;m)}(|z|^2) \; &= \; \frac{1}{\bar{n}+1} \frac{\Gamma(M+1)}{\Gamma(M+m+1)\Gamma(-M-m)} \frac{X^{-m}}{S_0^{(M;m)}(X)} \\ & G_{44}^{14}\left(\frac{\bar{n}}{\bar{n}+1} X \,\middle|\, \begin{array}{cccc} 1+M+m, & 0, & 0, & 1-2k; \\ 0; & m, & m, & 1-2k-m \end{array} \right). \end{aligned} \tag{75}$$

Proof 6.1: We start from the above expression for the reduced operator ρ_J:

$$Q_J^{(M;m)}(|z|^2) = \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n |\langle n;k|z, M; k; m\rangle|^2, \tag{76}$$

where the probability $P_n^{(M;m)}(|z|^2;k)$ of transition from the Fock state $|n;k\rangle$ to the EBS attached to the PHO $|z,M;k;m\rangle$ is

$$P_n^{(M;m)}(|z|^2;k) = |\langle n;k|z,M;k;m\rangle|^2 = \frac{1}{S_0^{(M;m)}(X)}\left[B_{n-m}^{(M;m)}\right]^2 X^{n-m}. \tag{77}$$

The square of coefficients can be expressed through the Pochhammer symbols:

$$\left[B_{n-m}^{(M;m)}\right]^2 = \Gamma(M+1)\frac{\Gamma(2k)}{\Gamma(M+m+1)\left[\Gamma(1-m)\right]^2\Gamma(2k-m)}\cdot\frac{(1)_n(2k)_n(-M-m)_n}{(-1)^n\left[(1-m)_n\right]^2(2k-m)_n}, \tag{78}$$

in order to express the Q- distribution function through the hypergeometric function ${}_pF_q(\ldots)$ and, finally, through Meijer's G - function.

$$\begin{aligned} Q_J^{(M;m)}(|z|^2) \;=\; & \frac{1}{\bar{n}+1}\frac{X^{-m}}{S_0^{(m)}(X)}\frac{\Gamma(M+1)\Gamma(2k)}{\Gamma(M+m+1)\left[\Gamma(1-m)\right]^2\Gamma(2k-m)} \\ & \cdot {}_4F_3\left(-M-m,1,1,2k;1-m,1-m,2k-m;-\frac{\bar{n}}{\bar{n}+1}X\right). \end{aligned} \tag{79}$$

From a practical point of view it is advantageous to express the hypergeometric function through the Meijer function by using the equation [13]

$$\begin{aligned} & {}_4F_3(-M-m,1,1,2k;1-m,1-m,2k-m;-\frac{\bar{n}}{\bar{n}+1}X) \\ & = \frac{\left[\Gamma(1-m)\right]^2\Gamma(2k-m)}{\Gamma(-M-m)\Gamma(2k)}G_{44}^{14}\left(\frac{\bar{n}}{\bar{n}+1}X\,\middle|\,\begin{matrix}1+M+m, & 0, & 0, & 1-2k; \\ 0; & & m, \; m, & 1-2k-m\end{matrix}\right). \end{aligned} \tag{80}$$

In this manner, we obtained the final results for the Q- distribution function of the EBSs for PHO. This ends the proof.

VI-2. P- (quasi-)distribution function

Now, let us calculate the diagonal representation of the "rotational" density operator in terms of EBSs, in order to find the P- (quasi-)distribution function.

Proposition 6.2: Let $|z,M;k;m\rangle$ *and* $\rho_J^{(M;m)}$ *be the EBSs, respectively, the "rotational" density operator for the PHO. Then the diagonal representation of the "rotational" density operator in terms of EBSs reads*

$$\rho_J^{(M;m)} = \int d\mu_J^{(m)}(z)|z,M;k;m\rangle P_J^{(M;m)}\left(|z|^2\right)\langle z,M;k;m|, \tag{81}$$

with the earlier determined integration measure $d\mu_J^{(m)}(z)$ *and the following* P- *(quasi) distribution function:*

$$P_J^{(M;m)}\left(|z|^2\right) = \frac{1}{\bar{n}}\frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}}\frac{G_{33}^{31}\left(\frac{\bar{n}+1}{\bar{n}}\frac{|z|^2}{1-|z|^2}\,\middle|\,\begin{matrix}M-1; & m, & 2k-1+m \\ 0, & 0, & 2k-1;\end{matrix}\right)}{G_{33}^{31}\left(\frac{|z|^2}{1-|z|^2}\,\middle|\,\begin{matrix}-M-1; & m, & 2k-1+m \\ 0, & 0, & 2k-1;\end{matrix}\right)}. \tag{82}$$

Proof 6.2: The rotational density operator $\rho_J^{(M;m)}$ for the excited states can be expressed in the Fock - vector representation by considering that the energy eigenvalues of an excited state are $E_{n+m,J} = E_{0,J} + \hbar\omega(n+m)$:

$$\rho_J^{(M;m)} = \frac{1}{Z_J^{(M;m)}(\beta)} \sum_{n=0}^{M} \left(\frac{\bar{n}}{\bar{n}+1}\right)^{n+m} |n+m;k\rangle\langle n+m;k|, \tag{83}$$

while the corresponding partition function is then

$$Z_J^{(M;m)}(\beta) = \sum_{n=0}^{M} \left(\frac{\bar{n}}{\bar{n}+1}\right)^{n+m} = (\bar{n}+1)\left(\frac{\bar{n}}{\bar{n}+1}\right)^{m} \left[1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}\right]. \tag{84}$$

On the one hand, the normalized rotational density operator for the excited states in the Fock-vector representation is

$$\rho_J^{(M;m)} = \frac{1}{(\bar{n}+1)\left[1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}\right]} \sum_{n=0}^{M} \left(\frac{\bar{n}}{\bar{n}+1}\right)^{n} |n+m;k\rangle\langle n+m;k|, \tag{85}$$

and on the other hand, in the EBSs representation, the same operator becomes

$$\begin{aligned} \rho_J^{(M;m)} &= \sum_{n=0}^{M} \left[B_{n-m}^{(M;m)}\right]^2 |n+m;k\rangle\langle n+m;k| \\ &\cdot \int_0^1 d\left(|z|^2\right) h_m^{(M)}(|z|^2;k) \frac{1}{(1-|z|^2)^M} \frac{1}{S_0^{(m)}(X)} (1-|z|^2)^M \\ &\left(\frac{\bar{n}}{\bar{n}+1}\right)^{n} P_J^{(M;m)}\left(|z|^2\right). \end{aligned} \tag{86}$$

By equalizing the right hand sides of the above two expressions for the density operator, and passing to the variable X simultaneously with the index change $n = s-1$, we are lead to the following integral equation:

$$\begin{aligned} &\int_0^\infty dX X^{s-1} \frac{h_m^{(M)}(X;k)}{(1+X)^2} \frac{P_J^{(M;m)}(X)}{S_0^{(M;m)}(X)} \\ &= \frac{1}{\bar{n}} \frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}} \frac{1}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^{s}} \frac{1}{\left[B_{s-1}^{(M;m)}\right]^2} \\ &= \frac{1}{\bar{n}} \frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}} \frac{1}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^{s}} \frac{1}{\Gamma(M+1)} \frac{\left[\Gamma(s)\right]^2 \Gamma(s+2k-1)\Gamma(M+2-s)}{\Gamma(s+m)\Gamma(s+2k-1+m)}. \end{aligned} \tag{87}$$

After the function change

$$G_J^{(M;m)}(X) \equiv \frac{h_m^{(M)}(X;k)}{(1+X)^2}\frac{P_J^{(M;m)}(X)}{S_0^{(M;m)}(X)}, \tag{88}$$

we get to the following Stieltjes moment problem:

$$\int_0^\infty dX X^{s-1} G_J^{(M;m)}(X)$$
$$= \frac{1}{\bar{n}}\frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}}\frac{1}{\left(\frac{\bar{n}+1}{\bar{n}}\right)^s}\frac{1}{\Gamma(M+1)}\frac{[\Gamma(s)]^2\,\Gamma(s+2k-1)\Gamma(M+2-s)}{\Gamma(s+m)\Gamma(s+2k-1+m)}. \tag{89}$$

The solution of such a problem is expressed through the Meijer G-function [7]:

$$G_J^{(M;m)}(X) = \frac{1}{\bar{n}}\frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}}\frac{1}{\Gamma(M+1)}G_{33}^{31}\left(\frac{\bar{n}+1}{\bar{n}}X \,\middle|\, \begin{matrix} -M-1; & m, & 2k-1+m \\ 0, & 0, & 2k-1; \end{matrix}\right). \tag{90}$$

In this manner, we obtain the above final expression of the P- (quasi-)distribution function.

Proposition 6.3: The P- distribution function must be normalized to unity, i.e., the following relation is fulfilled:

$$\int d\mu_J^{(m)}(z) P_J^{(M;m)}\left(|z|^2\right) = 1. \tag{91}$$

Proof 6.3: If we successively use the normalization condition of the density operator, the non-orthogonality relation, and the resolution of the unity operator for EBSs, we obtain

$$1 = \mathrm{Tr}\rho_J^{(M;m)} = \int d\mu_J^{(m)}(z')\langle z', M;k;m|\rho_J^{(M;m)}|z', M;k;m\rangle = \int d\mu_J^{(m)}(z) P_J^{(M;m)}\left(|z|^2\right), \tag{92}$$

where we have used the relation for the resolution of the identity operator and also the normalization relation for the EBSs.

By the corresponding substitutions, the last equality becomes

$$\int d\mu_J^{(m)}(z) P_J^{(M;m)}\left(|z|^2\right) = \frac{1}{\bar{n}}\frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}}\frac{1}{\Gamma(M+1)\Gamma(-M)}\cdot I_0^{(M)}(\bar{n}), \tag{93}$$

where with $I_0^{(M)}(\bar{n})$ we have denoted the integral involving the product of two Meijer G – functions:

$$I_0^{(M)}(\bar{n}) \equiv \int_0^\infty dX G_{33}^{13}\left(X \,\middle|\, \begin{matrix} M+1; & -m, & 1-2k-m \\ 0; & 1-2k, & 0 \end{matrix}\right)$$
$$G_{33}^{31}\left(\frac{\bar{n}+1}{\bar{n}}X \,\middle|\, \begin{matrix} -M-1; & m, & 2k-1+m \\ 0, & 0, & 2k-1; \end{matrix}\right). \tag{94}$$

This reduces to (as a particular case of Eq. (B4) from Appendix B)

$$I_0^{(M)}(\bar{n}) = \bar{n}\left[1 - \left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}\right]. \tag{95}$$

Therefore, the integral of the P- distribution function is really equal to unity. Now, let us calculate the thermal expectations of an observable using the rotational density operator in the EBSs representation. The corresponding formula is

$$\begin{aligned} \langle A\rangle_J^{(M;m)} \equiv \mathrm{Tr}\rho_J^{(M;m)}A &= \int d\mu_J^{(m)}(z')\langle z', M; k; m|\rho_J^{(M;m)}A|z', M; k; m\rangle \\ &= \int d\mu_J^{(m)}(z)P_J^{(M;m)}\left(|z|^2\right)\langle A\rangle_z^{(M;m)}. \end{aligned} \tag{96}$$

If we consider $A = N^s$ then we have

$$\begin{aligned} \langle N^s\rangle_J^{(M;m)} &= \int d\mu_J^{(m)}(z)P_J^{(M;m)}\left(|z|^2\right)\langle N^s\rangle_z^{(M;m)} \\ &= \frac{1}{\bar{n}}\frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}}\frac{1}{\Gamma(-M)\Gamma(M+1)}\sum_{l=0}^{s}\binom{s}{l}m^{s-l}\sum_{j=0}^{l}c_j^{(l)}I_j^{(M)}(\bar{n}), \end{aligned} \tag{97}$$

where the new integral is

$$\begin{aligned} I_j^{(M)}(\bar{n}) \equiv & \int_0^\infty dX G_{33}^{13}\left(X\,\middle|\,\begin{matrix} M+1, & -m, & 1-2k-m; \\ 0; & 1-2k, & j \end{matrix}\right) \\ & G_{33}^{31}\left(\frac{\bar{n}+1}{\bar{n}}X\,\middle|\,\begin{matrix} -M-1; & m, & 2k-1+m \\ 0, & 0, & 2k-1; \end{matrix}\right). \end{aligned} \tag{98}$$

By using the corresponding properties of the Meijer G-functions [7] and the notation $Y \equiv \bar{n}(\bar{n}+1)^{-1}$, we obtain successively

$$\langle N^s\rangle_J^{(M;m)} = \frac{1}{\frac{Y^{M+1}-1}{Y-1}}\sum_{l=0}^{s}\binom{s}{l}m^{s-l}\sum_{j=0}^{l}c_j^{(l)}Y^j\frac{d^j}{dY^j}\left[\frac{Y^{M+1}-1}{Y-1}\right], \tag{99}$$

or, again using Newton's binomial formula, finally, we obtain

$$\langle N^s\rangle_J^{(M;m)} = \frac{1}{\frac{Y^{M+1}-1}{Y-1}}\left(m + Y\frac{d}{dY}\right)^s\left[\frac{Y^{M+1}-1}{Y-1}\right] \equiv \langle N^s\rangle^{(M;m)}. \tag{100}$$

Consequently, the thermal expectations of the particle number operator do not depend on the rotational quantum number J, i.e., the rotational motion doesn't contribute to the thermal expectation value. This result was expected because the operator N was defined relative to changes in the number of vibrational quanta n: $N|n;k\rangle = n|n;k\rangle$, and, as a consequence, it does not change the rotational quantum number J.

The structure of the last equation imposes, naturally, the following notation for the finite sum, with the observation that $Y = Y(\bar{n})$:

$$S_M(Y) \equiv \sum_{n=0}^{M} Y^n = \frac{Y^{M+1} - 1}{Y - 1} \tag{101}$$

and $S'_M(\bar{n})$, $S''_M(\bar{n})$, ... as their first, second, ... derivative with respect to Y.

It can be seen that $S_M(Y)$ is independent on the excitation order m.

This allows us to obtain the final expression for the thermal expectation of the integer powers of the particle number operator:

$$\langle N^s \rangle^{(M;m)} = \frac{1}{S_M(Y)} \left(m + Y \frac{d}{dY} \right)^s S_M(Y). \tag{102}$$

Definition 6.1: The thermal analogue of the Mandel parameter was previously defined as [8, 14]

$$Q^{(M;m)} = \frac{\langle N^2 \rangle^{(M;m)} - \left[\langle N \rangle^{(M;m)} \right]^2}{\langle N \rangle^{(M;m)}} - 1. \tag{103}$$

By particularizing $s = 1$ and $s = 2$, for the EBSs this parameter becomes

$$Q^{(M;m)}(Y) = \frac{Y^2 \left[\frac{S''_M(Y)}{S_M(Y)} - \left(\frac{S'_M(Y)}{S_M(Y)} \right)^2 \right] - m}{Y \frac{S'_M(Y)}{S_M(Y)} + m} = \frac{Y^2 \frac{d}{dY} \left[\frac{d}{dY} \ln S_M(Y) \right] - m}{Y \left[\frac{d}{dY} \ln S_M(Y) \right] + m}. \tag{104}$$

So, the thermal analogue of Mandel's parameter is dependent on the mean number of particles $Y = Y(\bar{n})$, where $Y = \bar{n}/(\bar{n}+1) = \exp\left(-\frac{\hbar\omega}{k_B T} \right)$, and, ultimately, on the equilibrium temperature T. One can calculate the intervals where $Q^{(M;m)}(Y)$ is negative, zero, or positive, which correspond to sub-Poissonian, Poissonian, or super-Poissonian behavior of thermal states.

At the extreme equilibrium temperatures T the behavior of the thermal Mandel parameter is negative, as is shown in the table below.

Low temperatures	$T \to 0$	$Y \to 0$	$Q^{(M;m)} \to -1$	sub-Poissonian
High temperatures	$T \to \infty$	$Y \to 1$	$Q^{(M;m)} \to -1$	sub-Poissonian

So, the thermal BSs have a sub-Poissonian behavior at the extreme equilibrium temperatures.

VII. HARMONIC LIMIT OF THE EBSs FOR THE PHO

We organize this section in two steps. First, we apply the limit $m \to 0$ to some of the relations and characteristics $A^{(M;m)}$ regarding the EBSs for the PHO obtained in previous sections. As a result we obtain the corresponding relations for non-excited BSs for the PHO denoted by $A^{(M;0)}$. Second, we apply the harmonic limit to the obtained results $A^{(M;0)}$ for non-excited BSs for the PHO and recover the corresponding relations for the HO-1D, i.e., $A^{(HO-1D)}$ (see, e.g., [8]).

We must emphasize that the harmonic limit can be applied directly to the EBSs for the PHO, by using some limit properties of Meijer's G-functions, but the strategy which we have adopted seems to be much simpler. This approach can be presented as follows:

$$\begin{array}{ccccc} A^{(M;m)} & \rightarrow & A^{(M;0)} & \rightarrow & A^{(\text{HO-1D})} \\ (\text{EBSs for PHO}) & \lim\limits_{m\to\infty} & (\text{BSs for PHO}) & \lim\limits_{HO} & (\text{CSs for HO-1D}) \end{array}. \tag{105}$$

From the operational point of view, this means that

$$\lim_{HO}\left[\lim_{m\to 0} A^{(M;m)}\right] = \lim_{HO} A^{(M;0)} = A^{(HO-1D)}. \tag{106}$$

Definition 1. We call the harmonic limit $\lim\limits_{HO}$ *such a limit for which* $M \to \infty$ *and, simultaneously,* $|z|^2 \to 0$ *, but such that their product remains finite* $M|z|^2 = |\alpha|^2$ *, where* α *is a complex constant. The short notation of the harmonic limit is*

$$\lim_{HO}(\ldots) \equiv \lim_{\substack{M\to\infty \\ |z|^2\to 0 \\ M|z|^2\to|\alpha|^2}} (\ldots). \tag{107}$$

Implicitly, this approach may be a test of correctness for the obtained expressions.

VII-1. The non-excited BSs for the PHO (case $m = 0$)

Making use of the properties of reduction of order for the Meijer G-functions (see Appendix A, Eq. (A1)), we can calculate the sum $S_0^{(M;m)}(X)$ (Eq. (38)) for $m = 0$ [9]:

$$\begin{aligned} \lim_{m\to 0} S_0^{(M;m)}(X) \equiv S_0^{(M;0)}(X) &= \frac{1}{\Gamma(-M)} G_{33}^{13}\left(X \,\middle|\, \begin{matrix} M+1, & 0, & 1-2k; \\ 0; & 1-2k, & 0 \end{matrix}\right) \\ &= \frac{1}{\Gamma(-M)} G_{11}^{11}\left(X \,\middle|\, \begin{matrix} M+1 \\ 0 \end{matrix}\right) \\ &= \frac{1}{\Gamma(-M)}\Gamma(-M)(1+X)^M \\ &= \frac{1}{(1-|z|^2)^M}, \end{aligned} \tag{108}$$

so, we obtain $N_0^{(M)}(|z|^2) = 1$, i.e., just as for non-excited or usual BSs.

For the particular case of non-excited BSs ($m = 0$), Eq. (46) for the photon (boson) probability distribution leads to

$$\lim_{m\to 0} P_n^{(M;m)}(|z|^2;k) \equiv P_n^{(M;0)}(|z|^2;k) = \left(\begin{array}{c} M \\ n \end{array}\right) (|z|^2)^n (1-|z|^2)^{M-n}, \tag{109}$$

i.e., to the binomial distribution function.

Particularly, for $m = 0$, the Mandel parameter $Q_z^{(M;m)}$ for EBSs, Eq. (62) leads to the corresponding result for the usual BSs:

$$\lim_{m\to 0} Q_z^{(M;m)} \equiv Q_z^{(M;0)} = \frac{G_{33}^{13}\left(X \left| \begin{array}{c} \ldots,(0) \\ \ldots,2 \end{array}\right.\right)}{G_{33}^{13}\left(X \left| \begin{array}{c} \ldots,(0) \\ \ldots,1 \end{array}\right.\right)} - \frac{G_{33}^{13}\left(X \left| \begin{array}{c} \ldots,(0) \\ \ldots,1 \end{array}\right.\right)}{G_{33}^{13}\left(X \left| \begin{array}{c} \ldots,(0) \\ \ldots,0 \end{array}\right.\right)}. \tag{110}$$

From their mathematical structure, the obtained result is similar as that for the photon added coherent states for the PHO [6].

Let us calculate the above expression. By using the differential properties of the Meijer G-functions and also their specialized value, see Eqs. (A3) and (A7), we have [6]

$$\begin{aligned} G_{33}^{13}\left(X \left| \begin{array}{c} \ldots,(0) \\ \ldots,j \end{array}\right.\right) &\equiv G_{33}^{13}\left(X \left| \begin{array}{ccc} M+1, & 0, & 1-2k; \\ 0; & 1-2k, & j \end{array}\right.\right) \\ &= G_{22}^{12}\left(X \left| \begin{array}{cc} M+1, & 0; \\ 0; & j \end{array}\right.\right) = X^j \left(\frac{d}{dX}\right)^j G_{11}^{11}\left(X \left| \begin{array}{c} M+1 \\ j \end{array}\right.\right) \\ &= \frac{\Gamma(-M)\Gamma(M+1)}{\Gamma(M+1-j)} \frac{1}{(1-|z|^2)^M} \left(|z|^2\right)^j, \end{aligned} \tag{111}$$

and so, for the Mandel parameter for the BSs of the PHO, we obtain $Q_z^{(M;0)} = -|z|^2 < 0$, which shows that the BSs have a sub-Poissonian behavior.

The rotational Husimi Q- function $Q_J^{(M;m)}(|z|^2)$, Eq. (75) for $m = 0$ must tend to the corresponding function for the BSs:

$$\begin{aligned} \lim_{m\to 0} Q_J^{(M;m)}(|z|^2) &\equiv Q_J^{(M;0)}(|z|^2) \\ &= \frac{1}{\bar{n}+1} \frac{1}{\Gamma(-M)} \frac{\Gamma(-M)}{G_{11}^{11}\left(X \left| \begin{array}{c} 1+M \\ 0 \end{array}\right.\right)} G_{11}^{11}\left(\frac{\bar{n}}{\bar{n}+1} X \left| \begin{array}{c} 1+M \\ 0 \end{array}\right.\right) \\ &= \frac{1}{\bar{n}+1}\left(1 - \frac{1}{\bar{n}+1}|z|^2\right)^M \equiv Q^{(M;0)}\left(|z|^2\right). \end{aligned} \tag{112}$$

For $m = 0$ the rotational P- (quasi-)distribution function, Eq. (82) becomes

$$\lim_{m\to 0} P_J^{(M;m)}(|z|^2) \equiv P_J^{(M;0)}\left(|z|^2\right) = \frac{1}{\bar{n}}\frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}}\frac{G_{11}^{11}\left(\frac{\bar{n}+1}{\bar{n}}\frac{|z|^2}{1-|z|^2}\middle|\begin{array}{c}-M-1\\0\end{array}\right)}{G_{11}^{11}\left(\frac{|z|^2}{1-|z|^2}\middle|\begin{array}{c}-M-1\\0\end{array}\right)}$$

$$= \frac{1}{\bar{n}}\frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}}\frac{1}{\left(1+\frac{1}{\bar{n}}|z|^2\right)^{M+2}} \equiv P^{(M;0)}\left(|z|^2\right). \tag{113}$$

In these last two equations we used the last notation in order to evince that Husimi's Q- distribution function, as well as the P- (quasi-)distribution function of the BSs are independent of the rotational quantum number J.

By taking $m = 0$ in Eq. (104) we obtain the corresponding thermal analogue of Mandel's parameter for the usual BSs:

$$\lim_{m\to 0} Q^{(M;m)}(Y) \equiv Q^{(M;0)}(Y) = Y\left[\frac{S''_M(Y)}{S'_M(Y)} - \frac{S'_M(Y)}{S_M(Y)}\right]. \tag{114}$$

This equation can be expressed in the following manner, which is similar to the form presented in [4]:

$$Q^{(M;0)}(Y) = Y\frac{d}{dY}\left[\ln\left(\frac{d}{dY}\ln S_M(Y)\right)\right]. \tag{115}$$

These relations will be useful in order to calculate the harmonic limit.

VII-2. The harmonic limit

The usual or unexcited BSs have as the limit just the canonical coherent states (CSs) for the one-dimensional harmonic oscillator (HO-1D).

Proposition 8: At the harmonic limit, the BSs $|z, M; \lambda; 0\rangle$ tends to the canonical CSs for the HO-1D:

$$\lim_{HO}|z, M; \lambda; 0\rangle \equiv \lim_{\substack{M\to\infty \\ |z|^2\to 0 \\ M|z|^2\to|\alpha|^2}} |z, M; \lambda; 0\rangle \tag{116}$$

$$= \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n; \lambda = 0\rangle \equiv |\alpha; \lambda = 0\rangle_{HO-CSs}. \tag{117}$$

Proof 8. We begin from the asymptotic expression of the Euler Gamma function:

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x\left[1 + O\left(\frac{1}{x}\right)\right], \tag{118}$$

and we obtain successively

$$\left(\begin{array}{c} M \\ n \end{array}\right) = \frac{1}{n!}\frac{\Gamma(M+1)}{\Gamma(M+1-n)} \approx \frac{1}{n!}\frac{\Gamma(M)}{\Gamma(M-n)} = \frac{1}{n!}\sqrt{\frac{M-n}{M}}\left(\frac{M}{M-n}\right)^{M}\frac{1}{(M-n)^{-n}}\frac{e^{M-n}}{e^{M}}. \tag{119}$$

Having in mind that

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x} = e^{a}, \tag{120}$$

we obtain that the asymptotic for the binomial coefficient M is

$$\left(\begin{array}{c} M \\ n \end{array}\right) \approx \frac{1}{n!}(M-n)^{n}. \tag{121}$$

By applying the harmonic limits, we obtain

$$\begin{aligned}
&\lim_{HO}|z,M;\lambda;0\rangle \to \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}(M-n)^{\frac{n}{2}}\left(|z|^2\right)^{\frac{n}{2}}\left(1-|z|^2\right)^{\frac{M-n}{2}}|n;\lambda=0\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\left(1-\frac{n}{M}\right)^{\frac{n}{2}}\left(M|z|^2\right)^{\frac{n}{2}}\left(1-\frac{M|z|^2}{M}\right)^{\frac{M}{2}}\left(1-\frac{M|z|^2}{M}\right)^{-\frac{n}{2}}|n;\lambda=0\rangle \\
&= \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\underbrace{\left(\frac{1-\frac{n}{M}}{1-\frac{M|z|^2}{M}}\right)^{\frac{n}{2}}}_{\downarrow\ 1}\underbrace{\left(\sqrt{M|z|^2}\right)^{n}}_{\downarrow\ \alpha^n}\underbrace{\left(1-\frac{M|z|^2}{M}\right)^{\frac{M}{2}}}_{\downarrow\ e^{-\frac{1}{2}|\alpha|^2}}|n;\lambda=0\rangle \\
&= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n;\lambda=0\rangle \equiv |\alpha\rangle_{HO-1D},
\end{aligned} \tag{122}$$

and the proof is finished.

Thus, *the harmonic limit* $\lim\limits_{HO}$ leads the BSs to the HO-1D CSs in the same way as the binomial distribution tends to the Poisson distribution [15]:

$$\begin{aligned}
&P_n^{(M;0)}(|z|^2) \equiv |\langle n;\lambda|z,M;\lambda;0\rangle|^2 = \left(\begin{array}{c} M \\ n \end{array}\right)\left(|z|^2\right)^{n}(1-|z|^2)^{M-n} \\
&\to \frac{1}{n!}\left(1-\frac{n}{M}\right)^{n}\left(M|z|^2\right)^{n}\left(1-\frac{M|z|^2}{M}\right)^{M}\frac{1}{(1-|z|^2)^{n}} \\
&\to \frac{1}{n!}\left(|\alpha|^2\right)^{n}e^{-|\alpha|^2} = P_n^{(\mathrm{Poisson})}(|z|^2).
\end{aligned} \tag{123}$$

Proposition 9: The harmonic limit of the EBSs for the PHO $|z, M; k; m\rangle$ is

$$\lim_{HO} |z, M; k; m\rangle = \sqrt{\frac{\Gamma(2k)}{\Gamma(m+1)\Gamma(m+2k)} \frac{1}{{}_2F_2(m+1, 2k+m; 1, 2k; |\alpha|^2)}} (K_+)^m \left[\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n; k\rangle \right], \tag{124}$$

and, for $m = 0$, it leads to the canonical CSs for the HO-1D.

Proof 9. Beginning from the expression

$$\lim_{HO} |z, M; k; m\rangle = \frac{1}{\sqrt{\lim_{HO} S_0^{(M;m)}(X)}} \lim_{HO} \left[\sum_{n=0}^{M} B_n^{(M;m)} \left(\frac{z}{\sqrt{1-|z|^2}} \right)^n |n+m; k\rangle \right], \tag{125}$$

one can separately express the above limits:

$$\begin{aligned} \lim_{HO} B_n^{(M;m)} &= \sqrt{\frac{\Gamma(n+m+1)\Gamma(n+2k+m)}{\Gamma(n+1)\Gamma(n+2k)}} \sqrt{\lim_{HO} \binom{m}{n}} \\ &= \sqrt{\frac{\Gamma(n+m+1)\Gamma(n+2k+m)}{\Gamma(n+1)\Gamma(n+2k)}} \sqrt{\frac{1}{n!}(M-n)^n} \\ &= \sqrt{\frac{\Gamma(n+m+1)\Gamma(n+2k+m)}{\Gamma(n+1)\Gamma(n+2k)}} \frac{1}{\sqrt{n!}} \lim_{HO} \left(\sqrt{M}\right)^n, \end{aligned} \tag{126}$$

$$\begin{aligned} B_n^{(M;m)} \left(\frac{z}{\sqrt{1-|z|^2}} \right)^n &= \sqrt{\frac{\Gamma(n+m+1)\Gamma(n+2k+m)}{\Gamma(n+1)\Gamma(n+2k)}} \frac{1}{\sqrt{n!}} \lim_{HO} \left(\sqrt{M}\right)^n \left(\frac{z}{\sqrt{1-|z|^2}} \right)^n \\ &= \frac{1}{\sqrt{n!}} \lim_{HO} \left(\frac{\sqrt{M} z}{\sqrt{1-|z|^2}} \right)^n = \frac{1}{\sqrt{n!}} \alpha^n \lim_{HO} \left(\frac{1}{\sqrt{1-|z|^2}} \right)^n \\ &= \frac{1}{\sqrt{n!}} \alpha^n. \end{aligned} \tag{127}$$

On the other hand, the limit of the denominator is

$$\lim_{HO} S_0^{(M;m)}(X) = \frac{\Gamma(2k+m)\Gamma(m+1)}{\Gamma(2k)} \lim_{HO} {}_3F_2(-M, m+1, 2k+m; 1, 2k; -X). \tag{128}$$

In order to use a limit property of the hypergeometric functions ${}_pF_q(\ldots)$ (see, Appendix A, Eq. (A12)), we will perform the following transformations:

$$
\begin{aligned}
{}_3F_2(-M,m+1,2k+m;1,2k;-X) &= {}_3F_2\left(-M,m+1,2k+m;1,2k;-\frac{|z|^2}{1-|z|^2}\right)\\
&= {}_3F_2\left(-M,m+1,2k+m;1,2k;-\frac{1}{M}\frac{M|z|^2}{1-\frac{M|z|^2}{M}}\right)\\
&= {}_3F_2\left(-M,m+1,2k+m;1,2k;-\frac{1}{M}\frac{|\alpha|^2}{1-\frac{|\alpha|^2}{M}}\right)\\
&= {}_2F_2\left(m+1,2k+m;1,2k;|\alpha|^2\right). \qquad (129)
\end{aligned}
$$

Evidently, for the HO-CSs the parameter $\lambda = k$ is unimportant, so it can be taken to vanish. So, the BSs bridge the gab between the Fock state and the coherent states of the HO-1D by taking the above defined harmonic limit.

First, the integration measure (Eq. (45)) becomes

$$
\begin{aligned}
&\lim_{HO}\lim_{m\to 0} d\mu_m^{(M)}(z;k) = \lim_{HO} d\mu_0^{(M)}(z;k)\\
&= \frac{d\varphi}{2\pi}\lim_{HO}\frac{d(|z|^2)}{(1-|z|^2)^2}\lim_{HO} S_0^{(M;0)}(X)\lim_{HO}\frac{1}{\Gamma(M+1)}G_{33}^{31}\left(\frac{|z|^2}{1-|z|^2}\,\middle|\,\begin{matrix}-M-1;\\0;\end{matrix}\right)\\
&= \frac{d\varphi}{2\pi}\lim_{HO}\frac{d(|z|^2)}{(1-|z|^2)^2}\lim_{HO}\frac{1}{(1-|z|^2)^M}(M+1)\lim_{HO}\frac{1}{\Gamma(M+2)}G_{11}^{11}\left(\frac{|z|^2}{1-|z|^2}\,\middle|\,\begin{matrix}-M-1;\\0;\end{matrix}\right)\\
&= \frac{d\varphi}{2\pi}\lim_{HO}\frac{d(M|z|^2+|z|^2)}{(1-|z|^2)^{M+2}}\lim_{HO}\frac{1}{\left(1-\frac{M|z|^2}{M}\right)^M}\lim_{HO}\frac{1}{\Gamma(M+2)}G_{11}^{11}\left(\frac{1}{M}\frac{M|z|^2}{1-|z|^2}\,\middle|\,\begin{matrix}-M-1;\\0;\end{matrix}\right). \qquad (130)
\end{aligned}
$$

For the last limit we will use the limit property of the Meijer G-functions (see, Appendix, Eq. (A5)), so we obtain

$$
\begin{aligned}
\lim_{HO}\lim_{m\to 0} d\mu_m^{(M)}(z;k) &= \frac{d\varphi}{2\pi}d(|\alpha|^2)e^{-|\alpha|^2}G_{01}^{10}\left(|\alpha|^2\,|0\right) = \frac{d\varphi}{2\pi}d(|\alpha|^2)e^{-|\alpha|^2}e^{|\alpha|^2}\\
&= \frac{d\varphi}{2\pi}d(|\alpha|^2) = \frac{d^2z}{\pi} \equiv d\mu_0^{(HO-1D)}(z), \qquad (131)
\end{aligned}
$$

i.e., just the Lebesque measure.

The harmonic limit of the photon distribution (Eq. (46)) becomes

$$\begin{aligned}
\lim_{HO}\lim_{m\to 0} P_{n+m}^{(M;m)}(|z|^2;k) &= \lim_{HO}\lim_{m\to 0}\frac{1}{S_0^{(M;m)}(X)}\lim_{HO}\lim_{m\to 0}\left[B_{n-m}^{(M;m)}\right]^2 X^{n-m} \\
&= \lim_{HO}\frac{1}{S_0^{(M;0)}(X)}\lim_{HO}\left[B_n^{(M;0)}\right]^2 X^n \\
&= \lim_{HO}\frac{1}{\left(1-\frac{M|z|^2}{M}\right)^M}\lim_{HO}\frac{1}{n!}(M-n)^n\frac{1}{M^n}\left(\frac{M|z|^2}{1-|z|^2}\right)^n \\
&= \frac{1}{n!}e^{-|\alpha|^2}\left(|\alpha|^2\right)^n = P_n^{(HO-1D)}(|\alpha|^2), \qquad (132)
\end{aligned}$$

i.e., just the Poisson distribution for the HO-1D.

The harmonic limit of the particle number operator, Eq. (55) in the limit becomes

$$\begin{aligned}
&\lim_{HO}\lim_{m\to 0}\langle N^s\rangle_z^{(M;m)} \\
&= \lim_{HO}\lim_{m\to 0}\frac{\sum_{l=0}^{s}\binom{s}{l}m^{s-l}\left[\sum_{j=0}^{l}c_j^{(l)}G_{33}^{13}\left(X\,\middle|\,\begin{matrix}M+1, & -m, & 1-2k-m;\\ 0; & 1-2k, & j\end{matrix}\right)\right]}{G_{33}^{13}\left(X\,\middle|\,\begin{matrix}M+1, & -m, & 1-2k-m;\\ 0; & 1-2k, & 0\end{matrix}\right)} \\
&= \lim_{HO}\frac{\sum_{j=0}^{s}c_j^{(s)}G_{22}^{12}\left(X\,\middle|\,\begin{matrix}M+1, & 0;\\ 0; & j\end{matrix}\right)}{G_{22}^{12}\left(X\,\middle|\,\begin{matrix}M+1, & 0;\\ 0; & 0\end{matrix}\right)} = \lim_{HO}\frac{\sum_{j=0}^{s}c_j^{(s)}X^j\left(\frac{d}{dX}\right)^j G_{11}^{11}\left(X\,\middle|\,\begin{matrix}M+1\\ j\end{matrix}\right)}{G_{11}^{11}\left(X\,\middle|\,\begin{matrix}M+1\\ j\end{matrix}\right)} \\
&= \lim_{HO}\sum_{j=0}^{s}c_j^{(s)}\frac{\Gamma(M+1)}{\Gamma(M+1-j)}\left(|z|^2\right)^j = \lim_{HO}\sum_{j=0}^{s}c_j^{(s)}\binom{M+1}{j}j!\left(|z|^2\right)^j \\
&\approx \lim_{HO}\sum_{j=0}^{s}c_j^{(s)}(M+1-j)^j\left(|z|^2\right)^j = \sum_{j=0}^{s}c_j^{(s)}\left(M|z|^2\right)^j\lim_{HO}\left(1+\frac{1-j}{M}\right)^j \\
&= \sum_{j=0}^{s}c_j^{(s)}\left(|\alpha|^2\right)^j \equiv \langle N^s\rangle_\alpha^{(HO-1D)}. \qquad (133)
\end{aligned}$$

By particularizing for $s=1$ and 2, we obtain the harmonic limit of the Mandel parameter

(Eq. (62))

$$\begin{aligned}\lim_{HO}\lim_{m\to 0} Q_z^{(M;m)} &= \lim_{HO}\lim_{m\to 0} \frac{\langle N^2\rangle_z^{(M;m)} - \left(\langle N\rangle_z^{(M;m)}\right)^2}{\langle N\rangle_z^{(M;m)}} - 1 \\ &= \frac{\langle N^2\rangle_\alpha^{(HO-1D)} - \left(\langle N\rangle_\alpha^{(HO-1D)}\right)^2}{\langle N\rangle_\alpha^{(HO-1D)}} - 1 \\ &= \frac{|\alpha|^2 + |\alpha|^4 - |\alpha|^4}{|\alpha|^2} - 1 = 0. \end{aligned} \tag{134}$$

The same result may be obtained by using the result after Eq. (60):

$$\lim_{HO} Q_z^{(M;0)} = \lim_{HO} |z|^2 = \lim_{HO} \frac{M|z|^2}{M} = \lim_{HO} \frac{|\alpha|^2}{M} = 0. \tag{135}$$

This shows that the CSs of the HO-1D have a Poissonian behavior.

The harmonic limit of Husimi's function $Q_J^{(M;m)}(|z|^2)$ for the PHO is obtained by applying, as usual, first the limit $m \to 0$ and then the harmonic limit. So, the Husimi function for the usual or non-excited BSs is

$$\begin{aligned}\lim_{HO}\lim_{m\to 0} Q_J^{(M;m)}(|z|^2) &= \lim_{HO} Q_J^{(M;0)}(|z|^2) = \frac{1}{\bar{n}+1} \lim_{HO} \left(1 - \frac{\frac{1}{\bar{n}+1} M|z|^2}{M}\right)^M \\ &= \frac{1}{\bar{n}+1} e^{-\frac{1}{\bar{n}+1}|\alpha|^2} Q^{(HO-1D)}(|\alpha|^2). \end{aligned} \tag{136}$$

As can be seen, the rotational Husimi's Q- distribution function of the EBSs for the PHO is independent of $k = k(J)$, i.e., of the rotational quantum number J, only for the usual or non-excited BSs ($m = 0$).

By applying the harmonic limit, we get to the P- (quasi-)distribution function of the CSs of the HO-1D. For $m = 0$ the P- distribution function becomes

$$\begin{aligned}\lim_{m\to 0} P_J^{(M;m)}(|z|^2) \equiv P_J^{(M;0)}\left(|z|^2\right) &= \frac{1}{\bar{n}} \frac{1}{1 - \left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}} \frac{G_{11}^{11}\left(\frac{\bar{n}+1}{\bar{n}} \frac{|z|^2}{1-|z|^2} \,\middle|\, \begin{matrix} -M-1 \\ 0 \end{matrix}\right)}{G_{11}^{11}\left(\frac{|z|^2}{1-|z|^2} \,\middle|\, \begin{matrix} -M-1 \\ 0 \end{matrix}\right)} \\ &= \frac{1}{\bar{n}} \frac{1}{1 - \left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}} \frac{1}{\left(1 + \frac{1}{\bar{n}}|z|^2\right)^{M+2}} \equiv P^{(M;0)}\left(|z|^2\right). \end{aligned} \tag{137}$$

Then the harmonic limit becomes

$$\begin{aligned}\lim_{HO} P^{(M;0)}\left(|z|^2\right) &= \frac{1}{\bar{n}} \lim_{HO} \frac{1}{1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}} \lim_{HO} \frac{1}{\left(1+\frac{1}{\bar{n}}|z|^2\right)^{M+2}} \\ &= \frac{1}{\bar{n}} \lim_{HO} \frac{1}{\left(1+\frac{\frac{1}{\bar{n}}M|z|^2}{M}\right)^{M+2}} = \frac{1}{\bar{n}} \lim_{HO} \frac{1}{\left(1+\frac{\frac{1}{\bar{n}}|\alpha|^2}{M}\right)^{M+2}} \\ &= \frac{1}{\bar{n}} \lim_{HO} \frac{1}{\left(1+\frac{\frac{1}{\bar{n}}|\alpha|^2}{M}\right)^{M+2}} = \frac{1}{\bar{n}} e^{-\frac{1}{\bar{n}}|\alpha|^2} \equiv P^{(HO-1D)}\left(|z|^2\right). \end{aligned} \tag{138}$$

The harmonic limit for the thermal analogue of the Mandel parameter can be calculated by using the property of the limit operation:

$$\lim_{M\to\infty} Q^{(M;0)} = Y\frac{d}{dY}\left[\ln\left(\frac{d}{dY}\ln \lim_{M\to\infty} S_M(Y)\right)\right], \tag{139}$$

and having in mind that $Y < 1$, we obtain

$$\lim_{M\to\infty} S_M(Y) = \lim_{M\to\infty} \frac{Y^{M+1}-1}{Y-1} = \frac{1}{1-Y}. \tag{140}$$

So, finally, we obtain [2]

$$\lim_{M\to\infty} Q^{(M;0)} \equiv Q^{(\infty;0)} = \bar{n} = Q_{HO-1D}, \tag{141}$$

i.e., just the thermal analogue of the Mandel parameter for the HO-1D.

VIII. CONCLUSIONS

In this paper we have examined some properties of the ordinary and excited binomial states (BSs, respectively, EBSs), the later have been built by the repeated action of the lowering operator on the BSs. Although the definition of the usual version of the binomial states should not specify an orthogonal Fock-vector base, when we calculate the expectation values, there are determined from this basis. In this context we have paid our attention to the pseudoharmonic oscillator (PHO) basis, i.e., we have built the binomial states (BSs) for this oscillator using their Fock-vector basis $|n;k\rangle$.

Even if in the scientific literature ([1, 3]) it is specified that the BCs tend to the coherent states (CSs) at the harmonic limit, i.e., when $M \to \infty$, $|z|^2 \to 0$, but so that $M|z|^2 = |\alpha|^2$ (finite number), we have demonstrated that the ordinary (BSs) and also the excited binomial states (EBSs) have all the properties required of coherent states for the entire range of the complex variable, and so, the binomial states (ordinary and excited) belong to the family of coherent states. We consider this as the main result of the present paper, which has not yet appeared in the literature. Moreover, the BSs and EBSs have a sub-Poissonian behavior for small values of the argument $|z|^2$, respectively, for small extreme equilibrium temperatures (in the case of thermal states). So, we have shown that not only the non-excited binomial states (as stated in [1]), but also the excited binomial states are also sub-Poissonian.

We have also calculated different functions which characterize the two kind of binomial states (BSs and EBSs): the Q- and P- distribution functions, the Mandel parameter, and also the thermal analogue of the Mandel parameter. All these functions lead to the corresponding functions of the HO-1D, at the harmonic limit. This constitutes additional evidence for the obtained results. Moreover, the mathematical structure of these functions is similar to the structure of the corresponding functions for the CSs for the PHO and the photon added CSs for the PHO [8].

In all the calculations we have used Meijer's G-function formalism because of its simplicity and effectiveness in such kinds of problems [7]. Thereby implicitly the applicability area of these functions was expanded.

In conclusion, we consider that the present paper will be a small step in studying the properties both of excited binomial states and also in the properties of a pseudoharmonic oscillator. In this context a recent paper is of interest (where the pseudoharmonic oscillator appears with the name isotonic oscillator) [16].

APPENDIX A: SOME USEFUL PROPERTIES OF MEIJER'S G-FUNCTIONS AND HYPERGEOMETRIC FUNCTIONS [5, 17]:

If we use the notations: $\{a_p\} \equiv a_1, a_2, \ldots, a_n; a_{n+1}, a_{n+2}, \ldots, a_p$ and $\{b_q\} \equiv b_1, b_2, \ldots, b_m; b_{m+1}, b_{\mu+m}, \ldots, b_q$ then we point out the following properties of Meijer's G-functions:

- Reduction of order

$$G^{m,n+1}_{p+1,q+1}\left(X\,\middle|\,\begin{matrix} c, \{a_p\} \\ \{b_q\}, c\end{matrix}\right) = G^{m+1,n}_{p+1,q+1}\left(X\,\middle|\,\begin{matrix} \{a_p\}, & c \\ c, & \{b_q\}\end{matrix}\right) = G^{mn}_{pq}\left(X\,\middle|\,\begin{matrix} \{a_p\} \\ \{b_q\}\end{matrix}\right). \quad \text{(A1)}$$

- Symmetry and coefficient changes

$$G^{mn}_{pq}\left(X\,\middle|\,\begin{matrix} \{a_p\} \\ \{b_q\}\end{matrix}\right) = G^{nm}_{qp}\left(\frac{1}{X}\,\middle|\,\begin{matrix} \{1-b_q\} \\ \{1-a_p\}\end{matrix}\right); z^{\sigma} G^{mn}_{pq}\left(X\,\middle|\,\begin{matrix} \{a_p\} \\ \{b_q\}\end{matrix}\right) = G^{nm}_{qp}\left(X\,\middle|\,\begin{matrix} \{a_p+\sigma\} \\ \{b_q+\sigma\}\end{matrix}\right). \quad \text{(A2)}$$

- Differentiation

$$X^j \frac{d^j}{dX^j} G^{mn}_{pq}\left(X\,\middle|\,\begin{matrix} \{a_p\} \\ \{b_q\}\end{matrix}\right) = G^{m,n+1}_{p+1,q+1}\left(X\,\middle|\,\begin{matrix} 0, & \{a_p\} \\ \{b_q\}, & j\end{matrix}\right). \quad \text{(A3)}$$

- Classical Meijer's integral from two G functions

$$\int_0^\infty d\tau\tau^{\alpha-1}G_{u,v}^{s,l}\left(\tau w\,\middle|\,\begin{matrix} c_1,\ldots,c_l; & c_{l+1},\ldots,c_u \\ d_1,\ldots,d_s; & d_{s+1},\ldots,d_v\end{matrix}\right)G_{p,q}^{m,n}\left(\tau z\,\middle|\,\begin{matrix} a_1,\ldots,a_n; & a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m; & b_{m+1},\ldots,b_q\end{matrix}\right)$$
$$= \frac{1}{w^\alpha}\cdot G_{v+p,u+q}^{m+l,n+s}$$
$$\left(\frac{z}{w}\,\middle|\,\begin{matrix} a_1,\ldots,a_n, & 1-\alpha-d_1,\ldots,1-\alpha-d_s; & 1-\alpha-d_{s+1},\ldots,1-\alpha-d_v; & a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m, & 1-\alpha-c_1,\ldots,1-\alpha-c_l; & 1-\alpha-c_{l+1},\ldots,1-\alpha-c_u; & b_{m+1},\ldots,b_q\end{matrix}\right). \tag{A4}$$

- Limit operation

$$\lim_{c\to\infty}\frac{1}{\Gamma(1-c)}G_{p+1,q}^{m,n+1}\left(-\frac{X}{c}\,\middle|\,\begin{matrix} c,\{a_p\} \\ \{b_q\}\end{matrix}\right) = G_{p,q}^{m,n}\left(X\,\middle|\,\begin{matrix}\{a_p\} \\ \{b_q\}\end{matrix}\right). \tag{A5}$$

- Relation between Meijer's G-functions and hypergeometric functions:

$${}_pF_q(\{a_p\};\{b_q\};X) = \frac{\prod_{j=1}^p\Gamma(b_j)}{\prod_{i=1}^q\Gamma(a_i)}G_{p,q+1}^{1,p}\left(-X\,\middle|\,\begin{matrix} ; & \{1-a_p\} \\ 0; & \{1-b_q\}\end{matrix}\right). \tag{A6}$$

- Some particular values

$$G_{11}^{11}\left(X\,\middle|\,\begin{matrix} a \\ b\end{matrix}\right) = \Gamma(1-a+b)X^b(1+X)^{a-b-1}, \tag{A7}$$

$$G_{22}^{22}\left(X\,\middle|\,\begin{matrix} 1-a, & 1-b; \\ 0, & c-a-b;\end{matrix}\right) = \frac{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1\left(a,b;c;1-X\right). \tag{A8}$$

- Gauss hypergeometric function

$${}_2F_1\left(a,b;b+1;X\right) = bX^{-b}B_X(b,1-a), \tag{A9}$$

$${}_2F_1\left(1,b;c;X\right) = (c-1)X^{1-c}(1-X)^{-b+c-1}B_X(c-1,b-c+1). \tag{A10}$$

- Incomplete beta function

$$B_X(a,d) = \int_0^X t^{a-1}(1-t)^{d-1}dt. \tag{A11}$$

-Limit operation

$$\lim_{c\to\infty}{}_{p+1}F_q\left(c,\{a_p\};\{b_q\};\frac{X}{c}\right) =_p F_q(\{a_p\};\{b_q\};X). \tag{A12}$$

- Jacobi polynomials

$$G_{22}^{12}\left(X\left|\begin{matrix}a, & c;\\ b; & d\end{matrix}\right.\right) = \Gamma(b-a+1)\Gamma(d-c+1)X^{b}(X+1)^{a-b-1}P_j^{(-M-1;-j)}\left(\frac{1-X}{1+X}\right), \quad \text{(A13)}$$

$$P_j^{(a;b)}(-X) = (-1)^j P_j^{(b;a)}(X), \quad \text{(A14)}$$

$$P_j^{(a;-j)}(X) = \frac{1}{2^j}\frac{\Gamma(a+j+1)}{\Gamma(a+1)\Gamma(j+1)}(X+1)^j. \quad \text{(A15)}$$

APPENDIX B: SOME RELATIONS USED IN THE PRESENT PAPER

If we take $m = 0$ in the integration measure for EBSs (see Eq. (45)), we obtain successively:

$$\begin{aligned}
& d\mu_0^{(M)}(z;k) \\
= \; & \frac{1}{\Gamma(M+1)}\frac{1}{\Gamma(-M)}\frac{d\varphi}{2\pi}d(|z|^2)\frac{1}{(1-|z|^2)^2}\cdot\Gamma(M+2)\frac{1}{(1-|z|^2)^M}\Gamma(-M)\left(1-|z|^2\right)^{M+2} \\
= \; & (M+1)\frac{d\varphi}{2\pi}d(|z|^2). \qquad \text{(B1)}
\end{aligned}$$

The general integral $I_j^{(M)}(X)$ from Eq. (98) is

$$\begin{aligned}
& I_j^{(M)}(\bar{n}) \\
\equiv \; & \int_0^\infty dX G_{33}^{13}\left(X\left|\begin{matrix}M+1, & -m, & 1-2k-m;\\ 0; & 1-2k, & j\end{matrix}\right.\right) G_{33}^{31}\left(\frac{1}{Y}X\left|\begin{matrix}-M-1; & m, & 2k-1+m\\ 0, & 0, & 2k-1;\end{matrix}\right.\right) \\
= \; & G_{66}^{62}\left(\frac{1}{Y}\left|\begin{matrix}0, & -M-1; & m, & 2k-1+m, & 2k-1, & -j\\ -M-1, & m, & 2k-1+m, & 0, & 0, & 2k-1;\end{matrix}\right.\right) \\
= \; & G_{33}^{32}\left(\frac{1}{Y}\left|\begin{matrix}0, & -M-1; & -j\\ 0, & -M-1, & 0;\end{matrix}\right.\right) = G_{33}^{23}\left(Y\left|\begin{matrix}1, & M+2, & 1;\\ 1, & M+2; & 1+j\end{matrix}\right.\right) \\
= \; & \frac{1}{Y}G_{33}^{23}\left(Y\left|\begin{matrix}0, & M+1, & 0;\\ 0, & M+1, & j\end{matrix}\right.\right) = \frac{1}{Y}Y^j\left(\frac{d}{dY}\right)^j G_{22}^{22}\left(Y\left|\begin{matrix}M+1, & 0;\\ 0, & M+1;\end{matrix}\right.\right) \\
= \; & \Gamma(M+2)\Gamma(-M)\frac{1}{Y}Y^j\left(\frac{d}{dY}\right)^j {}_2F_1\left(1, M+2; 2; -\frac{1}{\bar{n}}\right) \\
= \; & \Gamma(M+2)\Gamma(-M)\frac{1}{Y}Y^j\left(\frac{d}{dY}\right)^j\left[\frac{1}{1-Y}B_{1-Y}(1, M+1)\right] \\
= \; & \Gamma(M+2)\Gamma(-M)\frac{1}{Y}Y^j\left(\frac{d}{dY}\right)^j\left[\frac{Y^{M+1}-1}{Y-1}\right]. \qquad \text{(B2)}
\end{aligned}$$

For $j = 0$ the general integral becomes

$$\begin{aligned} I_0^{(M)}(\bar{n}) &= G_{33}^{32}\left(\frac{1}{Y}\middle|\begin{matrix} 0, & -M-1; & -j \\ 0, & -M-1, & 0; \end{matrix}\right) \\ &= G_{22}^{22}\left(\frac{1}{Y}\middle|\begin{matrix} 0, & -M-1; \\ 0, & -M-1; \end{matrix}\right) \\ &= \Gamma(M+2)\Gamma(-M)_2F_1\left(1, M+2; 2; -\frac{1}{\bar{n}}\right) \\ &= \frac{\bar{n}}{M+1}\left[1-\left(\frac{\bar{n}}{\bar{n}+1}\right)^{M+1}\right]. \end{aligned} \tag{B3}$$

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