Nonrelativistic Reduction of the Bethe–Salpeter Wave Function for Fermion-Antifermion Bound States

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In this paper, the general form of the nonrelativistic Bethe–Salpeter wave function for fermion-antifermion bound states is presented. Together with the normalization condition and the corresponding Schrödinger equation, the nonrelativistic Bethe–Salpeter wave function can be calculated and can be used to compute the amplitudes for the processes involving bound states.

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I. INTRODUCTION

When decays or productions of a bound state are considered, the wave function, often the Bethe–Salpeter wave function [1, 2], of the bound state is needed to compute the corresponding amplitudes of the concerned processes. For describing heavy quarkonium the nonrelativistic bound-state picture can be adopted. The Bethe–Salpeter wave function can then be reduced to its nonrelativistic form [3–10] while the appropriate spin structure remains.

In Refs. [3–8], the nonrelativistic Bethe–Salpeter wave functions for the pseudoscalar state 0− and the vector state 1− are given. In Ref. [9], the nonrelativistic Bethe–Salpeter wave functions for the S wave state and P wave state are constructed in terms of the spin projection operators multiplied by the nonrelativistic bound-state wave function. A general form of the nonrelativistic Bethe–Salpeter wave function for the fermion-antifermion bound state is still absent. In Ref. [10], we have presented the general form of the nonrelativistic Bethe–Salpeter wave functions for bound states composed of one fermion and one scalar particle and of two scalar constituents. In this paper we will use the Bethe–Salpeter formalism employed in Ref. [10], which is the appropriate tool for describing bound state systems relativistically and has a solid basis in quantum field theory, to obtain the general form of the nonrelativistic reduction of the Bethe–Salpeter wave function for the fermion-antifermion bound state.

This paper is organized as follows. In Section II, the known relation between the Bethe–Salpeter equation and the Schrödinger equation is given for latter use. In Section III, the general form of the nonrelativistic reduction of the Bethe–Salpeter wave function is

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presented. The conclusion is in Section IV.

II. REDUCTION OF THE BETHE–SALPETER EQUATION TO THE SCHRÖDINGER EQUATION

The Bethe–Salpeter wave function for a fermion-antifermion bound state is defined by

$$\chi_P(x_1, x_2) = \langle 0 | T \Psi_1(x_1) \Psi_2(x_2) | P \rangle,$$

(1)

where $\Psi$ and $\Psi$ are the spinors for the fermion and antifermion fields, $T$ denotes the time ordering, $P$ is the momentum of the bound state, and all other indices are suppressed. Due to translational invariance the Fourier transformation can be written as

$$\chi_P(x_1, x_2) = e^{-iPX} \chi_P(x),$$

(2)

$$\chi_P(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \chi_P(p),$$

where

$$x = x_1 - x_2, \quad X = \eta_1 x_1 + \eta_2 x_2, \quad \eta_1 + \eta_2 = 1,$$

$$p = \eta_2 p_1 - \eta_1 p_2, \quad P = p_1 + p_2, \quad \eta_i = \frac{m_i}{m_1 + m_2}.$$  

(3)

The Bethe–Salpeter equation in momentum space reads [1, 2]

$$\chi_P(p) = S_F^1(p_1) \int \frac{d^3 p'}{(2\pi)^4} K(P, p, p') \chi_P(p') S_F^2(-p_2),$$

(4)

where $S_F^i(p_i)$ is the full fermion propagator. We will approximate the full propagators $S_F^i(p_i)$ by the free propagators [11, 12],

$$S_i(p_i) = \frac{i}{p_i - m_i + \imath \epsilon},$$

(5)

where $m_1$ and $m_2$ are interpreted as the effective masses for the fermion and antifermion.

The four-dimensional Bethe–Salpeter equation (4) faces problems in actual applications [12], thus the three-dimensional reductions of it are highly desirable [1, 2, 13–22]. In Ref. [23], it was shown that there exist infinite versions of the reduced Bethe–Salpeter equation. The most famous among them is the Salpeter equation [2] based on the assumption that the interaction between the constituents is instantaneous. In this paper, the Salpeter equation is employed.

We introduce the components of the relative momentum $p = p_\parallel + p_\perp$ parallel and perpendicular to the bound-state momentum $P$ by [4, 13, 24, 25]

$$\hat{P} = \frac{P}{M}, \quad M = \sqrt{P^2}, \quad p_\parallel = p \cdot \hat{P},$$

$$p = p_\parallel + p_\perp, \quad p_\parallel = p_1 \hat{P}, \quad p_\perp = p - p_1 \hat{P}, \quad d^4 p = dp_1 dp_3 dp_\perp,$$

(6)
where \( p_\parallel \) is the longitudinal part and \( p_\perp \) is the transverse part. In the rest frame of the bound state with momentum \( P = (M, 0) \), \( p_l = p_0 \), \( p_\parallel = (p_0, 0) \), and \( p_\perp = (0, \mathbf{p}) \). The projection operators can be written in covariant form \[24\],

\[
\Lambda_\pm^+(p_\perp) = \frac{\omega_i \pm H_i(p_\perp)}{2\omega_i}, \quad H_i(p_\perp) = \hat{P}(m_i - p_\perp),
\]

\[
\omega_i = \sqrt{m_i^2 + \mathbf{p}^2}, \quad \mathbf{\varpi} = -p_\perp^2,
\]

with the properties

\[
\Lambda_\pm^+(p_\perp)\Lambda_\pm^+(p_\perp) = 0, \quad \Lambda_\pm^+(p_\perp) + \Lambda_-^+(p_\perp) = 1,
\]

\[
\Lambda_\pm^+(p_\perp)\Lambda_\pm^+(p_\perp) = \Lambda_\pm^+(p_\perp), \quad H_i(p_\perp)\Lambda_\pm^+(p_\perp) = \pm \omega_i\Lambda_\pm^+(p_\perp) \tag{8}
\]

In this paper the covariant instantaneous approximation is employed \[24\], in which the approximated kernel is independent of the change of the longitudinal component of the relative momentum,

\[
K(P, p, p') \rightarrow K(p_\perp, p'_\perp) = iV(p_\perp, p'_\perp). \tag{9}
\]

It is a good approximation for a system composed of heavy and light constituents or of two heavy constituents which can move relativistically as a whole. It will reduce to the instantaneous approximation in the rest frame of the bound state.

Introduce the notation for later convenience

\[
\psi_P(p_\perp) = \int \frac{d\mathbf{p}}{2\pi} \chi_P(p), \quad \Gamma(p_\perp) = \int \frac{d^3p_\perp}{(2\pi)^3} V(p_\perp, p'_\perp) \psi_P(p'_\perp), \tag{10}
\]

where \( \psi_P(p_\perp) \) is the Salpeter wave function. Using Eqs. (5), (7), (9), and (10), the Bethe–Salpeter equation (4) becomes

\[
\chi(p) = \left[ \frac{\Lambda_\uparrow^+(p_\perp)}{\eta_1 M - p_l - \omega_1 + i\epsilon} + \frac{\Lambda_\downarrow(p_\perp)}{\eta_1 M + p_l + \omega_1 - i\epsilon} \right] \hat{P} \Gamma(p_\perp) \hat{P}
\]

\[
\times \left[ \frac{\Lambda_\downarrow(p_\perp)}{\eta_2 M - p_l - \omega_2 - i\epsilon} + \frac{\Lambda_\uparrow^-(p_\perp)}{\eta_2 M + p_l + \omega_2 + i\epsilon} \right] \tag{11}
\]

Performing the \( p_l \) integral in the Bethe–Salpeter equation (11) yields the Salpeter equation

\[
\psi_P(p_\perp) = \frac{\Lambda_\uparrow^+(p_\perp) \hat{P} \Gamma(p_\perp) \hat{P} \Lambda_\downarrow^-(p_\perp) - \Lambda_\downarrow(p_\perp) \hat{P} \Gamma(p_\perp) \hat{P} \Lambda_\uparrow^+(p_\perp)}{M - \omega_1 - \omega_2} \tag{12}
\]

Applying the energy projectors \( \Lambda_\uparrow^+(p_\perp) \) from the left hand side and \( \Lambda_\downarrow^-(p_\perp) \) from the right hand side to the Salpeter equation (12) leads to

\[
(M - \omega_1 - \omega_2)\psi^+_P(p_\perp) = \Lambda_\uparrow^+(p_\perp) \hat{P} \Gamma(p_\perp) \hat{P} \Lambda_\downarrow(p_\perp),
\]

\[
(M + \omega_1 + \omega_2)\psi^-_P(p_\perp) = -\Lambda_\downarrow(p_\perp) \hat{P} \Gamma(p_\perp) \hat{P} \Lambda_\uparrow^+(p_\perp). \tag{13}
\]
together with the constraints on the Salpeter wave function
\[ \psi_{\pm}^+(p_\perp) = \psi_{\pm}^-(p_\perp) = 0, \tag{14} \]
where \( \psi_{\pm}^\pm(p_\perp) = \Lambda^\pm_1(p_\perp) \psi_P(p_\perp) \Lambda^\pm_2(-p_\perp). \)

Let the bound state be normalized as \( \langle P|P' \rangle = (2\pi)^3 2 P_0 \delta(P - P'). \) Then the explicit form of the normalization condition for the Bethe–Salpeter wave function reads
\[
\int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \text{Tr} \left[ \chi(p) P^{\mu} \frac{d}{dp^\mu} \left( I(P, p, p') - K(P, p, p') \right) \chi(p') \right] = 2i M^2, \tag{15} \]
where \( I(P, p, p') = (2\pi)^4 \delta^4(p - p') S_{11}^{F-1}(p_1) S_{22}^{F-1}(-p_2). \) For the Salpeter wave function, the normalization condition (15) reduces to
\[
\int \frac{d^3 p}{(2\pi)^3} \text{Tr} \left\{ \hat{P} \psi(p_\perp) \hat{P} \Lambda^1_1(p_\perp) \psi(p_\perp) \Lambda^2_2(-p_\perp) - \hat{P} \psi(p_\perp) \hat{P} \Lambda^1_1(p_\perp) \psi(p_\perp) \Lambda^2_2(-p_\perp) \right\} = 2 M, \tag{16} \]
where \( \psi(p_\perp) = \gamma^0 \psi^+(p_\perp) \gamma^0. \)

For weakly bound states with \( M \approx m_1 + m_2 \) one has [26, 27]
\[
M - \omega_1 - \omega_2 \ll M + \omega_1 + \omega_2, \tag{17} \]
therefore the second term on the right side of the Salpeter equation (12) can be dropped. This leads to the reduced Salpeter equation [26, 27],
\[
(M - \omega_1 - \omega_2) \psi_P^-(p_\perp) = \Lambda^1_1(p_\perp) \hat{P} T(p_\perp) \hat{P} \Lambda^2_2(-p_\perp). \tag{18} \]
The reduction of the Salpeter equation to the reduced Salpeter equation (18) may be effected by imposing a further constraint:
\[
\Lambda^1_1(p_\perp) \psi_P(p_\perp) = \psi_P(p_\perp) \Lambda^2_2(-p_\perp) = 0. \tag{19} \]
The normalization condition Eq. (16) then becomes
\[
\int \frac{d^3 p}{(2\pi)^3} \text{Tr} \left\{ \hat{P} \psi(p_\perp) \hat{P} \Lambda^1_1(p_\perp) \psi(p_\perp) \Lambda^2_2(-p_\perp) \right\} = 2 M. \tag{20} \]

By neglecting all the spin degrees of freedom of the constituents, then restricting the whole formalism exclusively to positive-energy solutions, the spinless Salpeter equation can be obtained from Eq. (18):
\[
(M - \omega_1 - \omega_2) \phi(p_\perp) = \int \frac{d^3 p'_\perp}{(2\pi)^3} V(p_\perp, p'_\perp) \phi(p'_\perp), \tag{21} \]
which is obtained by simplifying the Bethe–Salpeter equation greatly, and can also be regarded as a relativistic generalization of the Schrödinger equation. By expanding \( \omega_i \) as follows:
\[
\omega_i = m_i + \frac{\omega_2}{2m_i} + \cdots, \tag{22} \]
Eq. (21) reduces to the Schrödinger equation [4, 22, 28–30]

\[ \left( \epsilon - \frac{\omega^2}{2\mu} \right) \phi(p_\perp) = \int \frac{d^3p_\perp'}{(2\pi)^3} V(p_\perp, p_\perp') \phi(p_\perp'), \]

where \( \epsilon = M - m_1 - m_2, \mu = m_1 m_2 / (m_1 + m_2). \)

**III. GENERAL FORM OF THE NONRELATIVISTIC BETHE–SALPETER WAVE FUNCTION**

**III-1. Nonrelativistic Bethe–Salpeter Wave function**

As discussed in Section II, by discarding the negative part, we can obtain from Eq. (11)

\[ \chi_P(p) = \frac{\Lambda^+_1(p_\perp) \hat{P} \Gamma(p_\perp) \hat{P} \Lambda^-_2(-p_\perp)}{(\eta_1 M + p_1 - \omega_1 + i\epsilon)(\eta_2 M - p_1 - \omega_2 + i\epsilon)}. \]

Using Eq. (18), the Bethe–Salpeter wave function is reduced to

\[ \chi_P(p) = \frac{\Lambda^+_1(p_\perp)(M - \omega_1 - \omega_2)i\psi(p_\perp)\Lambda^-_2(-p_\perp)}{(\eta_1 M + p_1 - \omega_1 + i\epsilon)(\eta_2 M - p_1 - \omega_2 + i\epsilon)}. \]

And the normalization condition for it is Eq. (20).

**III-2. \( \eta_P = (-1)^{j+1} \) case**

For a state with parity \( \eta_P = (-1)^{j+1} \) [22, 24], the general form of the Salpeter wave function reads

\[ \psi^j(p_\perp) = \gamma^5 \epsilon_{\mu_1 \cdots \mu_j} p^{\mu_2}_\perp \cdots p^{\mu_j}_\perp \left[ p^{\mu_1}_\perp (g_1 + \hat{P} g_2) + \gamma^{\mu_1}_\perp (g_3 + \hat{P} g_4) \right. \]

\[ + \hat{p}^{\mu_1}_\perp (g_5 + \hat{P} g_6) + \sigma^{\mu_1 \nu} \hat{p}_{\perp \nu} (g_7 + \hat{P} g_8) \left. \right], \]

where \( \gamma^\mu = \hat{p} \hat{P}^\mu, \gamma^{\mu}_\perp = \gamma^\mu - \gamma^\mu_{\parallel}, \sigma^{\mu \nu} = \left[ \gamma^\mu_{\parallel}, \gamma^\nu_{\parallel} \right], \hat{p}^\mu_{\perp} = p^\mu_{\perp} / \omega, g_i \equiv g_i(\omega). \) In the above equation, \( g_1, g_2, g_7, \) and \( g_8 \) are the main terms or large terms which are pure \( L = j \) wave components [31–33], while \( g_3, g_4, g_5, \) and \( g_6 \) are small terms which are relativistic corrections to \( g_1, g_2, g_7, \) and \( g_8 . \)

The constraints (14) on Eq. (26) produce

\[ g_3 = - \frac{(\omega_1 + \omega_2)\omega}{m_1 \omega_2 + m_2 \omega_1} g_7, \quad g_4 = \frac{(\omega_1 - \omega_2)\omega}{m_1 \omega_2 + m_2 \omega_1} g_8, \]

\[ g_5 = \frac{(\omega_1 - \omega_2)\omega}{m_1 \omega_2 + m_2 \omega_1} g_1 - \frac{(\omega_1 + \omega_2)\omega}{m_1 \omega_2 + m_2 \omega_1} g_7, \]

\[ g_6 = - \frac{(\omega_1 + \omega_2)\omega}{m_1 \omega_2 + m_2 \omega_1} g_2 + \frac{(\omega_1 - \omega_2)\omega}{m_1 \omega_2 + m_2 \omega_1} g_8. \]
The constraint \( \Lambda^{-1}_1 \psi \Lambda^+_2 = 0 \) yields

\[
g_1 = -\frac{\omega_1 + \omega_2}{m_1 + m_2} g_2, \quad g_8 = -\frac{\omega_1 \omega_2 + m_1 m_2 + \omega^2}{m_1 \omega_2 + m_2 \omega_1} g_7.
\]  (28)

In the nonrelativistic limit, it is straightforward from Eqs. (26), (27), and (28) to derive that, for the spin singlet state

\[
\psi_j^s(p_{\perp}) = \epsilon_{\mu_1 \cdots \mu_j} \hat{p}_{\perp}^{\mu_1} \cdots \hat{p}_{\perp}^{\mu_j} \Lambda^+_0 \gamma^5 f_s,
\]  (29)

and, for the spin triplet state

\[
\psi_j^t(p_{\perp}) = \epsilon_{\mu_1 \cdots \mu_j} \hat{p}_{\perp}^{\mu_1} \cdots \hat{p}_{\perp}^{\mu_j} \sigma^{\mu_1 \nu} \hat{p}_{\perp}^{\nu} \Lambda^+_0 \gamma^5 f_t,
\]  (30)

where \( \Lambda^+_0 = (1 + \hat{P})/2 \), \( f_s = 2 g_1 \), \( f_t = 2 g_7 \). In the nonrelativistic limit, the mixing between the spin singlet and spin triplet is neglected when the masses of the constituents are not equal. For the state 0−

\[
\psi^{0^-}(p_{\perp}) = \Lambda^0_0 \gamma^5 f,
\]  (31)

and, for the state 1+

\[
\psi_j^{1^+}(p_{\perp}) = \epsilon_{\mu_1 \cdots \mu_j} \hat{p}_{\perp}^{\mu_1} \cdots \hat{p}_{\perp}^{\mu_j} \Lambda^+_0 \gamma^5 f_s,
\]

\[
\psi_t^{1^+}(p_{\perp}) = \epsilon_{\mu} \sigma^{\mu \nu} \hat{p}_{\perp}^{\nu} \Lambda^+_0 \gamma^5 f_t.
\]  (32)

In the nonrelativistic limit, the normalization condition reduces to, for the spin singlet state

\[
\int \frac{d^3p_{\perp}}{(2\pi)^3} S_j^s f_s^2 = M,
\]  (33)

and, for the spin triplet state

\[
\int \frac{d^3p_{\perp}}{(2\pi)^3} [-S_j^s + S_j^t] f_t^2 = M,
\]  (34)

where

\[
S_j^s = \sum \mathcal{P}_{\mu_1 \cdots \mu_j \nu_1 \cdots \nu_j} \hat{p}_{\perp}^{\mu_1} \cdots \hat{p}_{\perp}^{\mu_j} \hat{p}_{\perp}^{\nu_1} \cdots \hat{p}_{\perp}^{\nu_j},
\]

\[
S_j^t = \sum \mathcal{P}_{\mu_1 \cdots \mu_j \nu_1 \cdots \nu_j} \gamma^{ \mu_1} \hat{p}_{\perp}^{\nu_1} \cdots \hat{p}_{\perp}^{\nu_j} \hat{p}_{\perp}^{\mu_1} \cdots \hat{p}_{\perp}^{\mu_j}.
\]  (35)

\( \mathcal{P} \) is defined in Eq. (A4).

If the Lorentz structure of the interaction kernel is a combination of a vector and scalar, \( V(p_{\perp}, p'_{\perp}) = V_s 1 \otimes 1 + V_v \gamma_\mu \otimes \gamma^\mu \), using Eqs. (18), (23), (29), and (30), the Schrödinger equation can be obtained for the singlet:

\[
\left( \epsilon - \frac{\omega^2}{2\mu} \right) S_j^s f_s(\omega) = \int \frac{d^3p'_{\perp}}{(2\pi)^3} (V_v - V_s) T_j^s f_s(\omega'),
\]  (36)
where
\[ \varpi' = \sqrt{-p_\perp^2}, \quad T_1^j = \sum \mathcal{P}_{\mu_1 \cdots \mu_j \nu_1 \cdots \nu_j} \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\mu_j} \hat{p}_\parallel^{\nu_1} \cdots \hat{p}_\parallel^{\nu_j}, \]
and, for the triplet
\[ \left( \epsilon - \frac{\varpi'^2}{2\mu} \right) [S_1^j + S_2^j] f_l(\varpi') = \int \frac{d^3p_\parallel}{(2\pi)^3} (V_v - V_s) [T_2^j - \hat{p} \cdot \hat{p}' T_3^j] f_l(\varpi'), \]
where
\[ T_2^j = \sum \mathcal{P}_{\mu_1 \cdots \mu_j \nu_1 \cdots \nu_j} \hat{p}_\perp^{\mu_1} \hat{p}_\parallel^{\nu_1} \hat{p}_\perp^{\mu_2} \cdots \hat{p}_\perp^{\mu_j}, \]
\[ T_3^j = \sum \mathcal{P}_{\mu_1 \cdots \mu_j \nu_1 \cdots \nu_j} g^{\mu_1 \nu_1} \hat{p}_\perp^{\mu_2} \cdots \hat{p}_\perp^{\mu_j} \hat{p}_\parallel^{\nu_2} \cdots \hat{p}_\parallel^{\nu_j}. \]

The Schrödinger equation (36) together with the normalization condition (33) determines the singlet wave function, and the Schrödinger equation (38) together with the normalization condition (34) determines the triplet wave function. Substituting Eqs. (29) and (30) into (25) yields the nonrelativistic Bethe–Salpeter wave function for the singlet state and for the triplet state with parity \( \eta_P = (-1)^j \).

**III-3.** \( \eta_P = (-1)^j \) case

For the state with parity \( \eta_P = (-1)^j \), the general form of the Salpeter wave function reads
\[
\psi^j(p_\perp) = \epsilon_{\mu_1 \cdots \mu_j} \hat{p}_\perp^{\mu_1} \cdots \hat{p}_\perp^{\mu_j} \left[ \hat{p}_\perp^{\mu_1}(g_1 + \hat{P} g_2) + \gamma_\perp^{\mu_1}(g_3 + \hat{P} g_4) + \hat{p}_\perp^{\mu_1}(g_5 + \hat{P} g_6) + \sigma^{\mu_1 \nu_1} \hat{p}_\parallel^{\nu_1}(g_7 + \hat{P} g_8) \right],
\]
where \( g_3, g_4 \) are pure \( L = j + 1 \) wave components, and \( g_5, g_6 \) are \( L = j + 1 \) states mixed with \( L - 1 \) wave components which are the main terms [31], while \( g_1, g_2, g_7, \) and \( g_8 \) are small terms which are relativistic corrections to \( g_3, g_4, g_5, \) and \( g_6 \).

The constraints on the Salpeter wave function (40), \( \Lambda_1^+ \psi \Lambda_2^+ = \Lambda_1^- \psi \Lambda_2^- = 0 \), produce
\[
g_1 = \frac{(\omega_1 + \omega_2)\varpi}{m_1 \omega_2 + m_2 \omega_1} (g_3 - g_5), \quad g_2 = \frac{(\omega_1 - \omega_2)\varpi}{m_1 \omega_2 + m_2 \omega_1} (g_4 - g_6),
g_7 = \frac{(\omega_1 - \omega_2)\varpi}{m_1 \omega_2 + m_2 \omega_1} g_3, \quad g_8 = \frac{(\omega_1 + \omega_2)\varpi}{m_1 \omega_2 + m_2 \omega_1} g_4.
\]

The constraint \( \Lambda_1^- \psi \Lambda_2^+ = 0 \) yields
\[
g_4 = -\frac{\omega_1 \omega_2 + m_1 m_2 + \varpi^2}{m_1 \omega_2 + m_2 \omega_1} g_3,
g_6 = -\frac{2 \varpi^2}{m_1 \omega_2 + m_2 \omega_1} g_3 - \frac{m_1 \omega_2 + m_2 \omega_1}{\omega_1 \omega_2 + m_1 m_2 + \varpi^2} g_5.
\]
Taking the nonrelativistic limit and using the wave function constraints (41) and (42), the general form of the wave function (40) reduces to, for the $L = j - 1$ wave state,
\[ \psi^j_S(p_\perp) = \epsilon_{\mu_1...\mu_j} \hat{p}_\perp^{\mu_j} ... \hat{p}_\perp^{\mu_1} \Lambda_0^+ \gamma_\perp \xi f_S, \] (43)
and, for the $L = j + 1$ wave state,
\[ \psi^j_D(p_\perp) = \epsilon_{\mu_1...\mu_j} \hat{p}_\perp^{\mu_j} ... \hat{p}_\perp^{\mu_1} \Lambda_0^+ \left( \hat{p}_\perp^{\mu_1} \hat{p}_\perp + \frac{j}{2j + 1} \gamma_\perp \right) f_D, \] (44)
where $f_S = 2g_3$, $f_D = 2g_5$. In the above formula (44) the $L - 1$ wave component is canceled by the second term in the brackets on the right side [31], and the pure $L = j + 1$ wave component is left. In the nonrelativistic limit, the mixing between the $L = j - 1$ wave component and the $L = j + 1$ wave component can be neglected [34]. For the state with $\eta_P = 0^+$,
\[ \psi^{0^+}(p_\perp) = \Lambda_0^+ \hat{p}_\perp f, \] (45)
and, for the state with $\eta_P = 1^-$,
\[ \psi^{1^-}(p_\perp) = \epsilon_\mu \Lambda_0^+ \gamma_\perp f_S, \]
\[ \psi^{1^-}(p_\perp) = \epsilon_\mu \Lambda_0^+ \left( \hat{p}_\perp^{\mu_1} \hat{p}_\perp + \frac{1}{3} \gamma_\perp \right) f_D. \] (46)

The normalization condition for the $L = j - 1$ wave state reads
\[ \int \frac{d^3p_\perp}{(2\pi)^3} (-S^j_2) f_S^2 = M, \] (47)
and for the $L = j + 1$ wave component it reads
\[ \int \frac{d^3p_\perp}{(2\pi)^3} \left[ \frac{(2j + 1)S^j_1 - j^2 S^j_2}{(2j + 1)^2} \right] f_D^2 = M. \] (48)

Using Eqs. (18), (23), (43), and (44), the Schrödinger equation for the $L = j - 1$ wave state is
\[ \left( \epsilon - \frac{\omega^2}{2\mu} \right) S^j_1 f_S(\omega) = \int \frac{d^3p'_1}{(2\pi)^3} (V_v - V_s) T^j_4 f_S(\omega'), \] (49)
where
\[ T^j_4 = \sum_{\mu_1...\mu_j\nu_1...\nu_j} \mathcal{P}_{\mu_1...\mu_j\nu_1...\nu_j} \hat{p}_\perp^{\mu_1} \hat{p}_\perp^{\mu_2} ... \hat{p}_\perp^{\nu_1} \hat{p}_\perp^{\nu_2} ... \hat{p}_\perp^{\nu_j}, \] (50)
and, for the $L = j + 1$ wave state it is
\[ \left( \epsilon - \frac{\omega^2}{2\mu} \right) S^j_1 f_D(\omega) \]
\[ = - \int \frac{d^3p'_1}{(2\pi)^3} (V_v - V_s) \left[ \frac{1 + 2j}{1 + j} \hat{p} \cdot \hat{p}' T^j_4 + \frac{j}{1 + j} T^j_4 \right] f_D(\omega'). \] (51)
The Schrödinger equation (49) together with the normalization condition (47) determines the $L = j - 1$ wave function, and the Schrödinger equation (51) together with the normalization condition (48) determines the $L = j + 1$ wave function. Substituting Eqs. (43) and (44) into (25), the nonrelativistic Bethe–Salpeter wave functions for the $L = j - 1$ state and for the $L = j + 1$ state can be obtained.

IV. CONCLUSION

In this paper, we have obtained the general form of the nonrelativistic Bethe–Salpeter wave function together with the normalization condition and the corresponding Schrödinger equation. Using the obtained nonrelativistic Bethe–Salpeter wave function, the amplitudes for the processes involving the bound state can be calculated.

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APPENDIX A: POLARIZATION TENSOR

It is known that the polarization tensor is totally symmetric, transverse, and traceless, i.e.,

$$\epsilon_{\mu_1 \mu_2 \cdots} = \epsilon_{\mu_2 \mu_1 \cdots}, \quad P^\mu_{\mu_1} \epsilon_{\mu_1 \mu_2 \cdots} = 0, \quad \epsilon^\mu_{\mu_1 \cdots} = 0. \quad (A1)$$

The usual spin-1 polarization vector $\epsilon^\mu(j_m)$ obeys the relations [9, 35]

$$P^\mu \epsilon_{\mu}(j_m) = 0, \quad \sum_{j_m} \epsilon^*_{\mu}(j_m) \epsilon_{\nu}(j_m) = P_{\mu \nu}, \quad P_{\mu \nu} \equiv -g_{\mu \nu} + \frac{P_\mu P_\nu}{M^2}. \quad (A2)$$

The polarization tensor $\epsilon^{\mu \nu}(j_m)$ is for a particle of spin-2 and it obeys

$$P^\mu \epsilon_{\mu \nu}(j_m) = 0, \quad \epsilon^{\nu \mu} = \epsilon^{\mu \nu}, \quad \epsilon^\mu_{\mu} = 0,$$

$$\sum_{j_m} \epsilon^*_{\mu \nu}(j_m) \epsilon_{\alpha \beta}(j_m) = \frac{1}{2} (P_{\mu \alpha} P_{\nu \beta} + P_{\nu \alpha} P_{\mu \beta}) - \frac{1}{3} P_{\mu \nu} P_{\alpha \beta}. \quad (A3)$$
For integer spin, the expression constructed in Refs. [36, 37] reads

\[
\mathcal{P}^{\mu_1 \cdots \mu_j \nu_1 \cdots \nu_j}(j, P) = \sum_{j_z} \epsilon^{\mu_1 \cdots \mu_j} \epsilon^{\nu_1 \cdots \nu_j} (j, P) = \left( \frac{1}{j!} \right)^2 \sum_{P(\mu)} \left[ \prod_{i=1}^{j} \mathcal{P}^{\mu_i \nu_i} + a_j^2 \mathcal{P}^{\mu_1 \mu_2} \mathcal{P}^{\nu_1 \nu_2} \prod_{i=3}^{j} \mathcal{P}^{\mu_i \nu_i} + \cdots \right. \\
+ a_j^2 \mathcal{P}^{\mu_1 \mu_2} \mathcal{P}^{\nu_1 \nu_2} \cdots \mathcal{P}^{\nu_{2r} \mu_{2r}} \mathcal{P}^{\mu_{2r} \nu_{2r}} \prod_{i=2r+1}^{j} \mathcal{P}^{\mu_i \nu_i} + \cdots \left. \\
+ \left\{ \begin{array}{ll} 
\frac{a_j^j}{2} \mathcal{P}^{\mu_1 \mu_2} \mathcal{P}^{\nu_1 \nu_2} \cdots \mathcal{P}^{\mu_{j-1} \nu_{j-1}} \mathcal{P}^{\nu_j \mu_j}, & \text{for even } j \\
\frac{a_j^{j-1}}{2} \mathcal{P}^{\mu_1 \mu_2} \mathcal{P}^{\nu_1 \nu_2} \cdots \mathcal{P}^{\mu_{j-2} \nu_{j-2}} \mathcal{P}^{\nu_{j-1} \mu_{j-1}} \mathcal{P}^{\mu_j \nu_j}, & \text{for odd } j 
\end{array} \right. \right]\right],
\]

where the sum is over all permutations of \( \mu \) and \( \nu \), and

\[
a_r^j = \left( \frac{-1}{2} \right)^r \frac{j!}{r!(j-r)!} \frac{(2j-r-1)!!}{(2j-1)!!}.
\]

In the above formula, \( n! \) gives the factorial of \( n \), \( n! = n(n-1)\cdots \), and \( n!! \) gives the double factorial of \( n \), \( n!! = n(n-2)\cdots \).

References