Model of the Neutrino Mass Matrix With $\delta = -\pi/2$ and $\theta_{23} = \pi/4$

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Experimental data have provided stringent constraints on the neutrino mixing parameters. In the standard parameterization the mixing angle $\theta_{23}$ is close to $\pi/4$. There is also evidence which shows that the CP violating phase is close to $-\pi/2$. We study the neutrino mass matrix reconstructed using this information and find several interesting properties. We show that a theoretical model based on the $A_4$ symmetry naturally predicts $\delta = -\pi/2$ and $\theta_{23} = \pi/4$ when the Yukawa couplings and scalar vacuum expectation values are real, reaching a $\mu - \tau$ exchange and CP conjugate symmetry limit. In this case CP violation solely comes from the complex group theoretical Clebsh-Gordan coefficients. The model also predicts $|V_{e2}| = 1/\sqrt{3}$, consistent with the data. With complex Yukawa couplings the values for $\delta$ and $\theta_{23}$ can significantly deviate away from the symmetry values $-\pi/2$ and $\pi/4$, respectively. But $|V_{e2}| = 1/\sqrt{3}$ is not altered. This matrix is an excellent lowest order approximation for theoretical model buildings of the neutrino mass matrix.

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Tremendous experimental progress has been made in obtaining information about the neutrino mixing parameters. The mixing angles in the Pontecorvo-Maki-Nagakawa-Sakata $V_{\text{PMNS}}$ matrix [1] are not always small [2–4]. In the standard parameterization [2, 5] for the three neutrino mixing commonly used [3, 4], the mixing angle $\theta_{23}$ is close to $\pi/4$, $\theta_{12}$ is large, $\theta_{13}$ is relatively small but away from zero, and also $s_{12}c_{13}$ is close to $1/\sqrt{3}$. Since the mixing angle $\theta_{13}$ is non-zero, the famous tri-bimaximal mixing [6] is ruled out. There is now evidence which shows that the CP violating phase $\delta$ is close to $-\pi/2$. This also implies that the tri-bimaximal mixing is in trouble, since it predicts $\delta = 0$. The phase $\delta$ is sometimes referred to as the Dirac phase, which shows up in neutrino oscillations. If neutrinos are Majorana particles, there are also new CP violating Majorana phases $\alpha_i$. There are many discussions about the implications for the data available emphasizing the particular values for $|\delta| = \pi/2$ and $\theta_{23} = \pi/4$ [8–10]. One of the commonly mentioned properties for this type of mixing is the so called maximal CP violation because $|\delta| = \pi/2$. This is, strictly speaking, an incorrect statement, because the value of the Dirac phase is parametrization dependent. For example, even though the absolute value of the Dirac phase is $\pi/2$ in the standard parametrization, in the original Kobayashi-Maskawa parametrization for quarks [11] it is not $\pi/2$ anymore. However, the special values for some of the mixing angles and the Dirac

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phase can still provide important information about the neutrino mass matrix and can guide theoretical model building to search for the underlying theory.

To this end, let us reconstruct the neutrino mass matrix assuming that neutrinos are Majorana particles with $\delta = -\pi/2$ and $\theta_{23} = \pi/4$. In the basis where the charged lepton mass matrix is already diagonalized, the neutrino mass matrix defined by the term giving neutrino masses in the Lagrangian $(1/2)\bar{\nu}_L m_\nu \nu^c_L$, has the following form:

$$m_\nu = V_{PMNS} \tilde{m}_\nu V_{PMNS}^T ,$$  \hspace{1cm} (1)

where $\tilde{m}_\nu = \text{diag}(m_1, m_2, m_3)$ with $m_i = |m_i| \exp(i \alpha_i)$. Here we have put the Majorana phase information in the neutrino masses. The standard form for $V_{PMNS}$ is given by

$$V_{PMNS} = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{12} s_{23} s_{13} e^{i \delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta} & s_{13} e^{-i \delta} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta} & -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta} & c_{23} c_{13} \end{pmatrix},$$  \hspace{1cm} (2)

where $c_{ij}$ and $s_{ij}$ are $\cos \theta_{ij}$ and $\sin \theta_{ij}$, respectively. With $\delta = -\pi/2$ and $\theta_{23} = \pi/4$, $m_\nu$ has the following form \[8, 10]\:

$$m_\nu = \begin{pmatrix} a & c + i \beta & -(c - i \beta) \\ c + i \beta & d + i \gamma & b \\ -(c - i \beta) & b & d - i \gamma \end{pmatrix},$$  \hspace{1cm} (3)

where

$$a = m_1 c_{12}^2 c_{13}^2 + m_2 s_{12}^2 c_{13}^2 - m_3 s_{13}^2, \quad b = -\frac{1}{2}(m_1(s_{12}^2 + c_{12}^2 s_{13}^2) + m_2(c_{12}^2 + s_{12}^2 s_{13}^2) - m_3 c_{13}^2),$$

$$c = -\frac{1}{\sqrt{2}}(m_1 - m_2)s_{12} c_{12} c_{13}, \quad d = \frac{1}{2}(m_1(s_{12}^2 - s_{12}^2 s_{13}^2) + m_2(c_{12}^2 - s_{12}^2 s_{13}^2) + m_3 c_{13}^2),$$

$$\beta = \frac{1}{\sqrt{2}} s_{13} c_{13}(m_1 c_{12}^2 + m_2 s_{12}^2 + m_3), \quad \gamma = -(m_1 - m_2)s_{12} c_{12} s_{13}.$$

(4)

Note that in the most general case, because of non-zero Majorana phases, the parameters $a$, $b$, $c$, $d$, $\beta$, and $\gamma$ are all complex.

The above matrix has a high level regularity pattern, implying that some underlying symmetry may be at work to produce it. Searching an underlying theory guided by symmetry principles may achieve this. Before doing this, however, it is worthwhile to understand more about the mass matrix in Eq. (3). An immediate question one may ask is that if, in general, the neutrino mass matrix in Eq. (3) always predicts $\delta = -\pi/2$ and $\theta_{23} = \pi/4$. The answer is negative. If $\delta = \pi/2$ and $\theta_{23} = \pi/4$, the neutrino mass matrix is given in a similar form as that in Eq. (3), but $\beta$ and $\gamma$ need to be multiplied by a “−” sign. Therefore without further information given, a general mass matrix in the form given by Eq. (3) can give $\delta = \pm \pi/2$ and $\theta_{23} = \pi/4$. Whether they predict $+\pi/2$ or $-\pi/2$, additional information needs to be provided. Moreover, if neutrinos have Majorana phases, the general form does not imply that $\delta$ and $\theta_{23}$ must take $\pm \pi/2$ and $\pi/4$, respectively, neither. This can be understood by studying the following quantity

$$m_\nu m^\dagger_\nu = V_{PMNS} \tilde{m}_\nu \tilde{m}^\dagger_\nu V_{PMNS}^\dagger.$$

(5)
The general form for the neutrino mass in Eq. (3) will give the “12” and “13” entries $A_{12,13}$ of $m_\nu m_\nu^\dagger$ as

$$A_{12} + A_{13} = -i2(a\beta^* + c\gamma^* - \beta d^* - \beta b^*)$$
$$= -(|m_1|^2 - |m_2|^2)s_{12}c_{13}(c_{23} - s_{23})$$
$$- (|m_1|^2c_{12}^2 + |m_2|^2s_{12}^2 - |m_3|^2)s_{13}c_{13}(c_{23} + s_{23})e^{-i\delta},$$
$$A_{12} - A_{13} = 2(ac^* + cd^* - cb^* + \beta \gamma^*)$$
$$= -(|m_1|^2 - |m_2|^2)s_{12}c_{13}(c_{23} + s_{23})$$
$$+ (|m_1|^2c_{12}^2 + |m_2|^2s_{12}^2 - |m_3|^2)s_{13}c_{13}(c_{23} - s_{23})e^{-i\delta}. \quad (6)$$

If the parameters in the set $P$: \{a, b, c, \beta, \gamma\}, are complex, for the above equations one can find solutions for other values of $\theta_{23}$ and $\delta$. Therefore the general neutrino mass matrix form does not imply that $\delta$ and $\theta_{23}$ must be $\pm \pi/2$ and $\pi/4$. If, however, the parameters in the set $P$ are all real, as long as $\sin \delta \neq 0$, one must have $s_{23} = c_{23}$ and $\delta = \pm \pi/2$, as can be seen from the above two equations. From Eq. (4) and Eq. (6), one also finds that all eigen-masses $m_i$ are real (the Majorana phases are zero or $\pi$). In this case the neutrino mass matrix can be rewritten as

$$m_\nu = \begin{pmatrix} A & C & -C^* \\ C & D^* & B \\ -C^* & B & D \end{pmatrix}, \quad (7)$$

with $A = a$, $B = b$, $C = c + i\beta$, and $D = d - i\gamma$. The most general $m_\nu$ can be written as [8]

$$m_\nu = \begin{pmatrix} e^{ip_1} & 0 & 0 \\ 0 & e^{ip_2} & 0 \\ 0 & 0 & e^{ip_3} \end{pmatrix} \begin{pmatrix} A & C & -C^* \\ C & D^* & B \\ -C^* & B & D \end{pmatrix} \begin{pmatrix} e^{ip_1} & 0 & 0 \\ 0 & e^{ip_2} & 0 \\ 0 & 0 & e^{ip_3} \end{pmatrix}, \quad (8)$$

where the phases $p_i$ are arbitrary.

All neutrino mass matrices which can be written in the above form, will predict $\delta = \pm \pi/2$, $\theta_{23} = \pi/4$ and all the eigen-masses are real. One can choose some particular values for $p_i$ to obtain forms of $m_\nu$ for convenience of analysis. For example the “---” sign for the “13” and “31” entries can be removed by choosing $p_1 = p_2 = 0$ and $p_3 = \pi$, the resultant matrix can be written in a more familiar form:

$$m_\nu = \begin{pmatrix} A & C & C^* \\ C & D^* & \tilde{B} \\ C^* & \tilde{B} & D \end{pmatrix}, \quad (9)$$

where $\tilde{B} = -B$.

The simplicity of the above mass matrix may serve as a good starting point to understand the possible underlying theory. If this has something to do with reality, one should not stay at the pure phenomenological level for analysis, but go further to study whether there are theoretical models which can obtain such a neutrino mass matrix in some consistently way. Several attempts for model buildings have been made [8, 9]. It has been shown in
Ref. [7] by Grimus and Lavoura that the above form of the mass matrix is symmetric under a transformation of $e \rightarrow e$, $\mu \rightarrow \tau$ exchange with a CP conjugation. We will refer to this as the Grimus-Lavoura symmetry (GLS). In this work we start with a simple model proposed earlier based on $A_4$ symmetry [13] to realize the tri-bimaximal neutrino mixing, and then modify it to allow a non-zero $\theta_{13}$ to find the conditions for having the GLS limit for the neutrino mass matrix with $\delta = -\pi/2$ and $\theta_{23} = \pi/4$ and how modifications may occur by explicit model studies. This model has an added bonus that [12, 13] $s_{12}c_{13} = V_{e2} = 1/\sqrt{3}$. There is also an interesting feature in this model that CP violation can be solely from the complexity of the relevant Clebsh-Gordan (C-G) coefficients in the GLS limit. We will refer this property as intrinsic CP violation.

In this model $A_4$ is serving as a family symmetry [13]. The Higgs sector is enlarged to have three Higgs fields, $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ (SM doublet), $\phi$ (SM doublet) and $\chi = (\chi_1, \chi_2, \chi_3)$ (SM singlet). Under the $A_4$, $\Phi$ and $\chi$ both transform as 3, and $\phi$ as 1. Three right-handed SM singlet neutrinos $\nu_R = (\nu^1_R, \nu^2_R, \nu^3_R)$ are introduced allowing the seesaw mechanism to be in effect. The standard left-handed leptons $l_L = (l^1_L, l^2_L, l^3_L)$, and standard right-handed charged leptons $(l^1_R, l^2_R, l^3_R)$, and $\nu_R$ transform as a 3, $(1, 1^n, 1^1)$ and 3, respectively. We refer the readers for more details on $A_4$ group properties to Refs. [8, 13, 14]. The Lagrangian responsible for the lepton mass matrix is

$$L = \frac{1}{\sqrt{2}}(\bar{l}_L \tilde{\Phi})_1 l^1_R + \lambda_\mu (\bar{l}_L \tilde{\Phi})_1 l^2_R + \lambda_\tau (\bar{l}_L \tilde{\Phi})_1 l^3_R + H.C.$$ 

where

$$\begin{align*}
(\bar{l}_L \tilde{\Phi})_1 l^1_R &= (\bar{l}^1_L \Phi_1 + \bar{l}^2_L \Phi_2 + \bar{l}^3_L \Phi_3) l^1_R, \\
(\bar{l}_L \tilde{\Phi})_1 l^2_R &= (\bar{l}^1_L \Phi_1 + \omega \bar{l}^2_L \Phi_2 + \omega^2 \bar{l}^3_L \Phi_3) l^2_R, \\
(\bar{l}_L \tilde{\Phi})_1 l^3_R &= (\bar{l}^1_L \Phi_1 + \omega^2 \bar{l}^2_L \Phi_2 + \omega \bar{l}^3_L \Phi_3) l^3_R.
\end{align*}$$

(11)

Here $\omega = \exp(i2\pi/3)$ and $\omega^2 = \exp(i4\pi/3)$ are the C-G coefficients of the $A_4$ group products.

If the vev structure is of the form $<\Phi_{1,2,3}> = v_\Phi$, $<\chi_1,3> = 0$, $<\chi_2> = v_\chi$, and $<\phi> = v_\phi$, one would obtain the charged lepton mass term as

$$\begin{pmatrix}
\bar{l}^1_L & \bar{l}^2_L & \bar{l}^3_L
\end{pmatrix} U_l \begin{pmatrix}
\sqrt{3} \lambda_\nu v_\Phi & 0 & 0 \\
0 & \sqrt{3} \lambda_\mu v_\Phi & 0 \\
0 & 0 & \sqrt{3} \lambda_\tau v_\Phi
\end{pmatrix} \begin{pmatrix}
l^1_R \\
l^2_R \\
l^3_R
\end{pmatrix}, \quad U_l = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}.$$ 

(12)

From the above, we can identify the charged lepton mass to be $m_i = \sqrt{3} \lambda_i v_\Phi$. The neutrino mass matrix has the seesaw form with

$$M = \begin{pmatrix}
0 & M_D \\
M_D^T & M_R
\end{pmatrix}, \quad M_R = \begin{pmatrix}
m & 0 & m_x \\
m & 0 & m_x \\
m_x & m_x & 0
\end{pmatrix},$$ 

(13)
where $M_D = \text{Diag}(1,1,1)\lambda_v v_\phi$, and $m_\chi = \lambda_\chi v_\chi$. From this one obtains the light neutrino mass matrix $M_\nu$ of the form given by

$$M_\nu = -M_D M_R^{-1} M_D = \begin{pmatrix} w & 0 & x \\ 0 & y & 0 \\ x & 0 & z \end{pmatrix},$$

(14)

where $w = z = (\lambda_v v_\phi)^2 m/(m^2 - m_\chi^2)$, $x = (\lambda_v v_\phi)^2 m_\chi/(m^2 - m_\chi^2)$, and $y = - (\lambda_v v_\phi)^2/m$.

The above model leads to the tri-bimaximal mixing which predicts $\theta_{13} = 0$. It had been the focus of $A_4$ symmetry studies for a few years [13, 15, 16]. But it is now ruled out because a non-zero $\theta_{13}$ has been measured. In this scheme, in order to obtain the tri-bimaximal mixing, the neutrino mass matrix with “11” and “33” entries equal is crucial. It has been pointed out [13] that a more natural form of the vev structure will lead to the “33” entry in the neutrino mass matrix to deviate from the “11” entry which leads to a non-zero $\theta_{13}$. To achieve this, for our purpose here, we will introduce two scalars $S_{1'}$ and $S_{1''}$ which are an SM singlet but transform as $1'$ and $1''$ under $A_4$. This results in two new terms for $M_R$ in the Lagrangian

$$Y_{S'}(\overline{\nu}_R \nu_{S'}^C)_{1'} S_{1'} + Y_{S''}(\overline{\nu}_R \nu_{S''}^C)_{1''} S_{1''} + H.C.$$ \hspace{1cm} (15)

After $S_{1',1''}$ develops a non-zero vev, $v_{S',S''}$, we have

$$M_R = \begin{pmatrix} m_1 & 0 & m_\chi \\ 0 & m_2 & 0 \\ m_\chi & 0 & m_3 \end{pmatrix},$$ \hspace{1cm} (16)

where $m_1 = m + Y_{S'} v_{S'} + Y_{S''} v_{S''}$, $m_2 = m + \omega^2 Y_{S'} v_{S'} + \omega Y_{S''} v_{S''}$, and $m_3 = m + \omega Y_{S'} v_{S'} + \omega^2 Y_{S''} v_{S''}$. The resulting light neutrino mass matrix $M_\nu$ no longer has $w = z$, but has

$$w = -\lambda_\nu^2 v_\phi^2 m_3/(m_1 m_2 - m_\chi^2), \quad z = -\lambda_\nu^2 m_1/(m_1 m_3 - m_\chi^2),$$ \hspace{1cm} (17)

and $x$ and $y$ are changed to

$$x = \lambda_\nu^2 v_\phi^2 m_\chi/(m_1 m_3 - m_\chi^2), \quad y = -\lambda_\nu^2 v_\phi^2/m_2.$$ \hspace{1cm} (18)

In the basis where the charged lepton mass matrix is diagonalized, the neutrino mass matrix becomes

$$m_\nu = U_l^\dagger M_\nu U_l^* = \frac{1}{3} \begin{pmatrix} w + 2x + y + z & w - \omega^2 x + \omega y + \omega z & w - \omega x + \omega y + \omega^2 z \\ w - \omega^2 x + \omega^2 y + \omega z & w + 2x + \omega y + \omega^2 z & w - x + y + z \\ w - \omega x + \omega y + \omega^2 z & w - x + y + z & w + 2\omega^2 x + \omega^2 y + \omega z \end{pmatrix}.$$ \hspace{1cm} (19)

Inserting $\omega = \exp(i2\pi/3)$ in the above, $m_\nu$ can be transformed into the form in Eq. (3) by redefined right-handed charged leptons. The parameters in the set $P_{A4} : \{w, x, y, z\}$ are in general complex, which will not always have $\delta = -\pi/2$ and $\theta_{23} = \pi/4$. One needs to work in the GLS limit, which can be realized if the parameters in the set $P_{A4}$ are all real. In
this case the complexity of the mass matrix is purely due to the $A_4$ group theoretical C-G coefficients $\omega$ and $\omega^2$. This is a case where CP violation is caused by the C-G coefficients, providing a concrete example of intrinsic CP violation.

Before we analyze the general features of the neutrino mass matrix with complex parameters in the set $P_{A_4}$, we would like to analyze the constraints on the model parameters to have the GLS limit, that is, to have $w, x, y, z$ to be real. The complexity of the parameters can appear in the Yukawa couplings, in the vevs, and also in places where $\omega^i$ appear in $m_4$. To make the Yukawa couplings and scalar vevs real, one can require the model Lagrangian to satisfy a generalized CP symmetry under which

$$\omega^A \rightarrow (\omega^1)^{CP}, (\omega^2)^{CP}, (\omega^3)^{CP}.$$  

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$$\omega^A \rightarrow (\omega^1)^{CP}, (\omega^2)^{CP}, (\omega^3)^{CP}.$$  

and all other fields transform the same as those under the usual CP symmetry. Here the superscript $CP$ in the above indicates that the fields are the usual $CP$ transformed fields.

The above transformation properties will transform relevant terms into their complex conjugate ones. Requiring the Lagrangian to be invariant under the above transformation dictates the Yukawa couplings to be real. The same requirement will dictate the scalar potential to forbid spontaneous CP violation and the vevs to be real. One, however, notices that the parameters $m_{2,3}$ are in general complex, even if the Yukawa couplings and the vevs of the scalar fields are made real, because of the appearance of $\omega^i$. To make them real to reach the GLS limit, it is therefore required that

$$\text{Im}(\omega^2Y^{S'}v^{S'} + \omega Y^{S''}v^{S''}) = \text{Im}(\omega Y^{S'}v^{S'} + \omega^2 Y^{S''}v^{S''}) = 0 .$$  

The above can be achieved by the absence of the scalar fields $S^{',''}$ in the theory or by setting $Y^{S'}v^{S'} = Y^{S''}v^{S''}$. If the vev structure of $\chi$ is fixed as given previously, absence of $S^{',''}$ will not have a phenomenologically acceptable mass matrix. Therefore, we will take the later possibility as an example of a GLS limit case to show some detailed features. In this case $M_\nu$ can be diagonalized by $V_\nu$ as the following:

$$M_\nu = V_\nu \hat{m}_\nu V_\nu^T , \quad V_\nu = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix} ,$$  

(22)

where $s = \sin \theta$ and $c = \cos \theta$. One obtains the mixing matrix to be

$$V_{\text{PMNS}} = U_1^T V_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix} c + s & 1 & c - s \\ c + ws & \omega^2 & \omega c - s \\ c + \omega^2 s & \omega & \omega^2 c - s \end{pmatrix} ,$$  

(23)

Normalizing the above mixing matrix to the standard parametrization in Eq. (2), one obtains

$$s_{12} = \frac{1}{\sqrt{2(1 + cs)}} , \quad s_{23} = \frac{1}{\sqrt{2}} , \quad s_{13} = \frac{(1 - 2cs)^{1/2}}{\sqrt{3}} .$$  

(24)
Here we have normalized $c_{ij}$ and $s_{ij}$ to be all positive. The neutrino eigen-masses are all real, but in general they can take positive or negative values depending on the values of $w$, $x$, $y$, and $z$. Note that the absolute values of the elements in the second column of $V_{PMNS}$ are all $1/\sqrt{3}$.

We now find the conditions for predicting $\delta = -\pi/2$ and $\delta = +\pi/2$. An easy way of doing this is to study the Jarlskog invariant quantity [18] $J = \text{Im}(V_{e1}V_{e2}^*V_{\mu1}V_{\mu2})$. Eqs. (2) and (23) give

$$J = c_{13}^2 s_{12} c_{13}^{} s_{23}^{} s_{13}^{} \sin \delta = -\frac{1}{6\sqrt{3}} (c^2 - s^2),$$

which leads to

$$\delta = \frac{\pi}{2} \times \begin{cases} -1, & \text{if } c^2 > s^2, \\ +1, & \text{if } s^2 > c^2. \end{cases}$$

Note that $J$ is not zero, implying CP violation, which is caused by the complexity of the C-G coefficients. Eq. (23) can be transformed into the standard parameterization by multiplying the $V_{PMNS}$ on the right and left by diagonal matrices $P_r = \text{diag}(1,1,i)$ and $P_l = \text{diag}(1,(\omega^2 c - s)/|\omega^2 c - s|, (\omega c - s)/|\omega c - s|)$, respectively. $P_l$ does not have a physical effect because it can be absorbed by a redefinition of the right-handed charged leptons. The physical effect of $P_r$ is to change the sign of $m_3$.

Let us now compare the experimental data with the model predictions for the mixing angles and the CP violating phase. There are several global fits of neutrino data [3, 4]. The latest fit gives the central values, 1σ errors, and the 2σ ranges as the following [3]:

<table>
<thead>
<tr>
<th>NH</th>
<th>$\delta/\pi$</th>
<th>$s_{12}^2$</th>
<th>$s_{13}^2/10^{-2}$</th>
<th>$s_{23}^2$</th>
<th>2σ region</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.41$^{+0.55}_{-0.44}$</td>
<td>0.323 $\pm$ 0.016</td>
<td>2.26 $\pm$ 0.12</td>
<td>0.567$^{+0.032}_{-0.124}$</td>
<td>0.0 $\sim$ 2.0</td>
</tr>
<tr>
<td>IH</td>
<td>1.48 $\pm$ 0.31</td>
<td>0.323 $\pm$ 0.016</td>
<td>2.29 $\pm$ 0.12</td>
<td>0.573$^{+0.025}_{-0.039}$</td>
<td>0.00 $\sim$ 0.09 $&amp;$ 0.86 $\sim$ 2.0</td>
</tr>
<tr>
<td>2σ range</td>
<td>0.292 $\sim$ 0.357</td>
<td>0.20 $\sim$ 2.50</td>
<td>0.414 $\sim$ 0.623</td>
<td>0.435 $\sim$ 0.621</td>
<td></td>
</tr>
</tbody>
</table>

Here $NH$ and $IH$ indicate neutrino mass hierarchy patterns of normal hierarchy and inverted hierarchy, respectively. In the model above, adjusting the values, $w$, $x$, $y$, and $z$, both the NH and IH mass patterns can be obtained. There is a strong hint that the Dirac phase should be close to $3\pi/2$ (or equivalently $-\pi/2$). Therefore one should take the parameter space so that $c^2 > s^2$. The value $-\pi/2$ predicted in the model is in agreement with IH within the 1σ range. Although for the NH case $\delta$ is outside of the 1σ range, there no problem with the 2σ range. For $s_{23}$, the model predicts $s_{23}^2 = 0.5$. This value is outside of the 1σ range for both the NH and IH cases. However, they are, again, in agreement with the data within 2σ.

In the model $s_{13} = (1 - 2cs)^{1/2}/\sqrt{3}$ is not predicted. But one can use information from $s_{13}$ to fix $cs = 0.497 \pm 0.018$ to predict $s_{23}^2 = 0.334 \pm 0.004$ for both the NH and IH cases. This is in agreement with the data within 1σ. Note that $V_{e2}^2 = (s_{12}c_{13})^2 = 1/3$. It agrees with the data within 1σ. It is remarkable that the neutrino mixing matrix in this model with just one free parameter can be in reasonable agreement with the data.
This may be a hint that it is the form for the mixing matrix, at least as the lowest order approximation, that a underlying theory is producing.

If \( w, x, y, \) and \( z \) are allowed to be complex, the GLS is explicitly broken, there are modifications to the mixing angles. There is an additional source for CP violation other than the intrinsic one from complexity of the C-G coefficient, and also the mixing angles will be modified. The eigen-masses will contain Majorana phases. Detailed analysis of how to diagonalize the mass matrix has been discussed in Ref. [12]. In general this model does not always predict \( \delta = \pm \pi/2 \) and \( \theta_{23} = \pi/4 \). The mixing matrix can be, in general, written as

\[
V_{\text{PMNS}} = U_l^\dagger V_\rho V_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix}
  c + se^{i\rho} & 1 - ce^{i\rho} & c + ce^{i\rho} \\
  c + \omega se^{i\rho} & \omega^2 - s & \omega ce^{i\rho} \\
  c + \omega^2 se^{i\rho} & \omega & \omega^2 ce^{i\rho}
\end{pmatrix},
\]

where \( V_\rho \) is a diagonal matrix \( \text{diag}(1, 1, e^{i\rho}) \) with \( \tan \rho = \text{Im}(xw^* + x^* z)/\text{Re}(xw^* + x^* z) \). It is interesting that the phase \( \rho \) does not show up in \( J \), which is still \(-c^2 - s^2/6\sqrt{3}\). This implies that the CP violation related to neutrino oscillation is still purely due to intrinsic CP violation. The mixing angles and the Dirac phase \( \delta \) are all modified with

\[
s_{12} = \frac{1}{\sqrt{2(1 + cs \cos \rho)}} \quad s_{23} = \frac{(1 + cs \cos \rho + \sqrt{3cs \sin \rho})^{1/2}}{\sqrt{2(1 + cs \cos \rho)}}, \quad s_{13} = \frac{(1 - 2cs \cos \rho)^{1/2}}{\sqrt{3}},
\]

and

\[
\sin \delta = \left(1 + \frac{4c^2 s^2 \sin^2 \rho}{(c^2 - s^2)^2}\right)^{-1/2}(1 - \frac{3c^2 s^2 \sin^2 \rho}{(1 + cs \cos \rho)^2})^{-1/2} \times \left\{ \begin{array}{ll}
-1, & \text{if } c^2 > s^2 \\
+1, & \text{if } s^2 > c^2.
\end{array} \right.
\]

In this case, the new parameter \( \rho \) can be used to improve the agreement of the model with the data. In both the NH and IH cases, \( \delta \) and \( s_{23} \) can be brought into agreement with the data at the 1\( \sigma \) level. To see how this can be done, as an example, we take the largest value of \( cs \) so that \( s_{13} \) takes its lower 1\( \sigma \) allowed value, and then varying \( \cos \rho \) to obtain the upper 1\( \sigma \) allowed value. This fixes \( cs \) and \( \cos \rho \) to be 0.468 and 0.992, respectively. With these values, \( s_{23} \) and \( \delta \) are determined to: 0.534 and 1.426\( \pi \), respectively. These values are in agreement with data at the 1\( \sigma \) level. When more precise experimental data become available, the model with complex model parameters can be distinguished from that with the parameters being all real and other models.

In summary we have shown that neutrino mass matrix reconstructed with \( \delta = -\pi/2 \) and \( \theta_{23} = \pi/4 \) has several interesting properties. We find that a theoretical model based on the \( A_4 \) symmetry naturally realizes the GLS limit and predicts such a neutrino mixing pattern together with the prediction \(|V_{e2}| = 1/\sqrt{3}\). In this model, CP violation can come solely from the complex group theoretical C-G coefficients if the neutrino Majorana phases are zero or \( \pi \). This model fits the experimental data very well and can be taken as the lowest order neutrino mass matrix for future theoretical model buildings. If there are additional sources of CP violation other than those intrinsically existing in the C-G coefficients, the CP violating phase \( \delta \) and the mixing angle \( \theta_{23} \) can be away from \(-\pi/2 \) and \( \pi/4 \). The
models discussed can fit the data within $1 \sigma$. Future improved experimental data will be able to further test the model and provide more hints for the underlying theory of neutrino mixing.

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