Quantum Phases of a Deformed Itinerant Electron Model in One Dimension

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A one-dimensional half-filled deformed itinerant electron model is studied analytically, and the corresponding quantum phase diagram is obtained via the bosonization and the renormalization group techniques in the weak-coupling regime. The introduction of a small deformation parameter \( \lambda \ll 1 \) softens the induced two- and three-body interaction strengths. The properties of the model are determined by the critical value \( \lambda_c = \sqrt{\pi U/8} \), with \( U \) being the on-site repulsion caused by the release of no double occupancy due to the addition of \( \lambda \).

When \( \lambda > \lambda_c \), the ground state consists of the triplet-superconducting (TS) correlations and insulating spin-density-wave (SDW) and bond-charge-density-wave (BOW) phases. When \( \lambda \leq \lambda_c \), the superconductivity (SC) correlations disappear, and the insulating phases persist. The anisotropy (\( J_\perp \neq J_z \)) leads to the removal of critical spin correlations. The result indicates that the physics of the deformed version is not equivalent to that of the conventional \( t-J \) model. The study provides an important insight into the \( t-J \) model and its extended versions.

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I. INTRODUCTION

The study of strongly correlated electronic systems continues to be at the center of interest in condensed matter, and the \( t-J \) model is a paradigm for theoretical studies. Especially, since the discovery of high-\( T_c \) superconductivity, the \( t-J \) model has been attracting much attention, because it was suggested to be a suitable candidate for realizing the resonating valence bond scenario [1]. In two dimensions, even for the most elementary models that are believed to capture some of the physics of the cuprates, the problem remains challenging, but much can be learned from the study of one-dimensional (1D) counterparts of such models. In addition, some common properties of 2D systems can be shared by some 1D systems. Basic features characterizing the high-\( T_c \) materials are essential to 1D systems and are well understood here [2, 3]. The success of the theory in 1D is due to the availability of powerful exact approaches. As a result, it is instructive to study the one-dimensional version. This further helps to understand the high-dimensional physics.

To some extent, the 1D \( t-J \) model could be regarded as a descendant of the Hubbard model in the large on-site repulsion limit. Or equivalently, the strong-coupling limit of the Hubbard model can map onto the weak-coupling limit of the \( t-J \) model. Nevertheless,
the $t$-$J$ model cannot inherit the 1D Hubbard model in the whole parameter space. For example, the Hubbard model is integrable for arbitrary $U$, while the $t$-$J$ model is exactly solved only at two supersymmetric points [4–6]. Moreover, these integrable points are not located in the weak coupling regime but in the strong coupling regime. The reason for this is that, in contrast with other 1D integrable systems, the Hilbert space of the $t$-$J$ model is highly constrained. That is, a state with double occupancy is entirely excluded. The $t$-$J$ model Hamiltonian reads

$$H = \mathcal{P}\{-t \sum_{j,\alpha} (c_{j,\alpha}^\dagger c_{j+1,\alpha} + \text{h.c.}) + J \sum_j S_j \cdot S_{j+1}\} \mathcal{P}.$$  

(1)

Here $\mathcal{P} = \prod_j (1 - n_{j, \uparrow} n_{j, \downarrow})$ is a projection operator that enforces the constraint of no doubly occupied sites. The 1D $t$-$J$ model cannot be exactly solved at a generic parameter, so the analytical investigations of the $t$-$J$ model have been an arduous task even in the 1D case. In order to get over the difficulties associated with the no-double-occupancy constraint, a variety of analytical tools, e.g., the slave fermion and slave boson methods [7, 8], the supersymmetric Hubbard operator method [9], and the Spl(1,2) algebra method [10], have been developed to handle the $t$-$J$ model. Besides, some numerical calculations were applied to obtain a complete picture of the $t$-$J$ model [11–13]. On the other hand, various extended $t$-$J$ models have also been studied widely [14–23].

The seminal work by Chen and Wu provided an alternative idea to treat the $t$-$J$ model theoretically [24]. That is, a kind of deformed Hubbard operator method was proposed so that the 1D $t$-$J$ model can be handled by the bosonization technique in the weak-coupling regime. When the deformation parameter $\lambda$ changes from zero to unit, the no double occupancy constraint is carried out continuously in a controllable way. In the limit $\lambda = 1$, there is no essential qualitative change in the phase diagram of the model. Namely, the Hamiltonian of the model evolves adiabatically with $\lambda$. This idea has already been applied to other 1D models [25, 26], but at the same time, Ma et al. pointed out that the deformed model does not relate to the conventional $t$-$J$ model [26]. On the other hand, due to anisotropic exchange, e.g., spin-orbit coupling [27], the $t$-$J_{\perp}$-$J_z$ model attracts a great deal of attention. The model Hamiltonian is given by

$$H = -t \sum_{j,\alpha} \overline{c}_{j,\alpha}^\dagger \overline{c}_{j+1,\alpha} + \text{h.c.} + \frac{J_\perp}{2} \sum_j (S_j^+ S_{j+1}^- + \text{h.c.}) + J_z \sum_j S_j^z S_{j+1}^z,$$

(2)

where the Hubbard operators, $\overline{c}_{j,\alpha} = c_{j,\alpha} (1 - n_{j, -\alpha})$, equally realize the constraint with no double occupancy [28]. It is seen that the $t$-term in Eq. (1) is no longer a simple hopping of electrons. The extra two- and three-electron interactions are induced, with strengths of the same order magnitude as the hopping amplitude $t$. A couple of cousins of the model (2), the $t$-$J_z$ and $t$-$J_{\perp}$ forms were discussed separately [24, 29].

In Ref. [24], Chen and Wu discussed the case of half filling, and the corresponding charge mode is gapless. So the given phase diagram only describes the low-energy spin dynamics. On the other hand, the release of no double occupancy is bound to affect the original electron-electron interactions, which was not taken into account. In this paper, we
study a deformed version of the 1D $t$-$J_{\perp}$-$J_z$ model (see the following). We focus on the half-filled band, positive interactions, and the weak coupling regime ($0 \leq J_{\perp}, J_z, U \ll t \equiv 1, \lambda \ll 1$). By improving the deformed Hubbard operator approach, the overall quantum phase diagram, including both the charge and spin physics, is obtained by the bosonization and renormalization group analysis. The properties of the model are determined by the critical value $\lambda_c = \sqrt{\pi U/8}$. When $\lambda > \lambda_c$, the ground state consists of the triplet-superconducting (TS) correlations and insulating spin-density-wave (SDW) and bond-charge-density-wave (BOW) phases. When $\lambda \leq \lambda_c$, the SC correlations disappear. The anisotropy ($J_{\perp} \neq J_z$) leads to the removal of critical spin correlations. The result indicates that the physics of the deformed $t$-$J$ model is not equivalent to that of the conventional $t$-$J$ model. Our study provides an important insight into the $t$-$J$ model and its extended versions.

II. DEFORMED ITINERANT ELECTRON MODEL AND BOSONIZATION

We replace the Hubbard operators in Eq. (2) by the deformation operator
\begin{equation}
\tilde{c}_{j,\alpha} = c_{j,\alpha}(1 - \lambda n_{j,-\alpha}),
\end{equation}
with the deformation parameter $0 < \lambda \leq 1$. For $\lambda = 1$, the Hubbard operators are recovered. However, for $0 < \lambda < 1$ there is a nonzero probability to allow leakage into states with double occupancy. In case double occupation is allowed, the on-site $U$ should be considered. As a result, the 1D deformed version of the $t$-$J_{\perp}$-$J_z$ model Eq. (2) is described by the Hamiltonian
\begin{equation}
H = -t \sum_{j,\alpha} (\tilde{c}^\dagger_{j+1,\alpha} \tilde{c}_{j,\alpha} + h.c.) + U \sum_j n_{j,\uparrow} n_{j,\downarrow} \\
+ \frac{J_{\perp}}{2} \sum_j (S_j^+ S_{j+1}^- + h.c.) + J_z \sum_j S_j^z S_{j+1}^z.
\end{equation}

In order to understand the many-body physics well, it is helpful to rewrite the model Hamiltonian (4) explicitly in terms of the second quantized form as
\begin{equation}
H = H_{tUJ} + H_{t\lambda} + H_{t\lambda^2},
\end{equation}
with
\begin{align}
H_{tUJ} &= -t \sum_{j,\alpha} (c^\dagger_{j,\alpha} c_{j+1,\alpha} + h.c.) + U \sum_j n_{j,\uparrow} n_{j,\downarrow}, \\
+ J_{\perp} \sum_j (S_j^+ S_{j+1}^- + h.c.) + J_z \sum_j S_j^z S_{j+1}^z, \\
H_{t\lambda} &= t\lambda \sum_{j,\alpha} [c^\dagger_{j,\alpha} c_{j+1,\alpha} (n_{j+1,-\alpha} + n_{j+1,-\alpha}) + h.c], \\
H_{t\lambda^2} &= -t\lambda^2 \sum_{j,\alpha} (c^\dagger_{j,\alpha} c_{j+1,\alpha} n_{j,-\alpha} n_{j+1,-\alpha} + h.c).
\end{align}
The $H_{tUJ}$ component describes the usual $t$-$U$-$J$ model [30, 31]. The $H_{t\lambda}$ and $H_{t\lambda^2}$ terms denote the induced two-body and three-body interactions, respectively. The minus in the $H_{t\lambda^2}$ term represents attraction, it makes up for the excessive repulsion in the $H_{t\lambda}$ term. Apparently, the induced two- and three-body interactions are proportional to $\lambda$ and $\lambda^2$, respectively. When $0 < \lambda \ll 1$, all the induced interactions can be treated as perturbations and an effective field-theoretical approach may be safely used.

In the 1D case, the bosonization approach is a powerful tool to analyze the Hamiltonian (4). The 1D low-energy excitation spectrum is linearized around two Fermi points $\pm k_F$, and the fermion operator $c_{j,\alpha}$ is replaced by

$$c_{j,\alpha} \rightarrow -i^j \psi_{-j,\alpha} + i^j \psi_{+j,\alpha},$$

where the fields $\psi_{-j,\alpha}$ and $\psi_{+j,\alpha}$ describe left-moving and right-moving particles, respectively, and are assumed to be smooth on the scale of the lattice spacing $a$. This assumption allows one to introduce the continuum fields $\psi_{-\alpha}(x)$ and $\psi_{+\alpha}(x)$ by

$$\psi_{-j,\alpha}(x) \rightarrow \sqrt{a}\psi_{-\alpha}(x), \quad \psi_{+j,\alpha}(x) \rightarrow \sqrt{a}\psi_{+\alpha}(x).$$

The left-moving and right-moving fermion fields $\psi_{-\alpha}(x)$ and $\psi_{+\alpha}(x)$ are defined as [32]

$$\psi_{-\alpha}(x) = \frac{1}{\sqrt{2\pi a}} e^{-i\sqrt{4\pi}\varphi_{-\alpha}(x)},$$

$$\psi_{+\alpha}(x) = \frac{1}{\sqrt{2\pi a}} e^{i\sqrt{4\pi}\varphi_{+\alpha}(x)}.$$  \hspace{1cm} (11)

In terms of the chiral bosonic fields $\varphi_{\pm\alpha}(x)$, one introduces a pair of conjugate scalar fields $\phi_{\alpha}(x)$ and $\theta_{\alpha}(x)$

$$\phi_{\alpha}(x) = \varphi_{+,\alpha}(x) + \varphi_{-,\alpha}(x),$$

$$\theta_{\alpha}(x) = \varphi_{-,\alpha}(x) - \varphi_{+,\alpha}(x).$$  \hspace{1cm} (12)

These two boson operators obey the commutation relation

$$[\phi_{\alpha}(x), \theta_{\alpha'}(x')] = i\pi\delta_{\alpha\alpha'}\text{sgn}(x-x').$$

We introduce the linear combinations

$$\phi_{c/s}(x) = \frac{\phi_{+}(x) \pm \phi_{-}(x)}{\sqrt{2}}, \quad \theta_{c/s}(x) = \frac{\theta_{+}(x) \pm \theta_{-}(x)}{\sqrt{2}},$$

which describe the charge ($\mu = c$) and spin ($\mu = s$) degrees of freedom, respectively. Under the Bogoliubov transformation, we rescale the charge and spin fields as

$$\phi_{\mu} \rightarrow \sqrt{K_{\mu}}\phi_{\mu}, \quad \theta_{\mu} \rightarrow \frac{1}{\sqrt{K_{\mu}}}\theta_{\mu}.$$  \hspace{1cm} (16)

Using formulas (11)–(16), the bosonized version of the Hamiltonian (4) describing the low-energy states acquires the following form:

$$H = H_c + H_s + H_{cs},$$

\hspace{1cm} (17)
with

\[
H_c = v_c \int dx \left\{ \frac{1}{2} (\partial_x \phi_c)^2 + (\partial_x \theta_c)^2 \right\} + \frac{g_u}{2\alpha \pi^2} \cos(\sqrt{8\pi K_c} \phi_c),
\]

\[
H_s = v_s \int dx \left\{ \frac{1}{2} (\partial_x \phi_s)^2 + (\partial_x \theta_s)^2 \right\} + \frac{g_\perp}{2\alpha \pi^2} \cos(\sqrt{8\pi K_s} \phi_s),
\]

\[
H_{cs} = -\frac{g_{cs}}{2\alpha \pi^2} \int dx \cos(\sqrt{8\pi K_c} \phi_c) \cos(\sqrt{8\pi K_s} \phi_s).
\]

(18) \hspace{1cm} (19) \hspace{1cm} (20)

Above we have defined

\[
K_\mu \simeq 1 + \frac{g_\mu}{4\pi t}, \quad v_\mu = \frac{v_F}{K_\mu}, \quad (\mu = c, s),
\]

(21)

where \(v_c\) and \(v_s\) are the renormalized Fermi velocities for the charge and spin channels, respectively. At half filling, up to the lowest order in \(U\), \(J_\perp\), and \(J_z\), the Fermi velocity \(v_F = 2ta\), and the small bare coupling constants read (hereafter, \(t \equiv 1\))

\[
g_c = g_u = \frac{8\lambda^2}{\pi} - J_2 - J_\perp - U,
\]

(22)

\[
g_s = \frac{8\lambda^2}{\pi} - 3J_z + J_\perp + U,
\]

(23)

\[
g_\perp = \frac{8\lambda^2}{\pi} + J_z - J_\perp + U,
\]

(24)

\[
g_{cs} = -\frac{4\lambda^2}{\pi} - \frac{J_z}{2}.
\]

(25)

In Eqs. (18)--(20), the single cosine terms describe Umklapp scattering and backward scattering, respectively, and they can give rise to the charge-gap and spin-gap instabilities. In contrast, the double cosine term delineates the charge-spin coupling. Due to the \(g_{cs}\) term with a nonzero conformal spin, the dynamics of the model becomes much more involved. Fortunately, in the weak coupling limit it possesses higher scaling dimensionality. Therefore, we may first neglect the component \(H_{cs}\), just as in the case of other works [33–37].

**III. THE RG ANALYSIS**

With the charge-spin separation hypothesis, the initial Hamiltonian (4) is mapped onto the continuum theory of two decoupled quantum SG models Eqs. (18)--(19). The deformed \(t-J_\perp-J_z\) model can be analyzed by the “g-ology” technique [38–42]. To this end, we examine the relative importance of these couplings by the RG analysis in the weak-coupling regime. The low-energy properties of the system are described by pairs of RG equations for the effective coupling constants \(\Gamma_i\) [30, 43]

\[
\frac{d\Gamma_c(l)}{dl} = -\Gamma_u^2(l), \quad \frac{d\Gamma_u(l)}{dl} = -\Gamma_c(l)\Gamma_u(l),
\]

(26)

\[
\frac{d\Gamma_\perp(l)}{dl} = -\Gamma_{\perp}^2(l), \quad \frac{d\Gamma_\perp(l)}{dl} = -\Gamma_s(l)\Gamma_\perp(l),
\]

(27)
where we have performed the scale transformation of the cutoff $a \rightarrow ae^{d_{\text{l}}}$, with $l$ the length scale. The dimensionless running coupling constants $\Gamma_i(0) = g_i/2\pi ta$. The solutions to these two pairs of scaling equations construct the RG flow diagram shown in Fig. 1.

![Diagram of renormalization-group flow](image)

**FIG. 1:** The renormalization-group flow diagram. The arrows represent the direction of flow with increasing length scale. WC and SC correspond to the weak-coupling and strong-coupling regimes, respectively. $\Delta_{s/c}$ denotes the spin (charge) gap. The infrared behavior is determined by the direction of the RG flow with increasing length scale.

Using the initial values of the coupling constants given by (22)–(24), we observe that, due to the SU(2) symmetry, the trajectories in the charge channel are exactly along the separatrix $g_u = g_c$. Consequently, for

$$J_\perp + J_z^2 + U - \frac{8\lambda^2}{\pi} > 0$$

there exists a gap in the charge excitation spectrum ($\Delta_c > 0$). The RG flows scale toward the strong-coupling (SC) sector, $\Gamma_u(l \rightarrow \infty) \rightarrow -\infty$, and the field $\phi_c$ gets ordered with the vacuum expectation value

$$\langle \phi_c \rangle = 0.$$  

While for

$$J_\perp + J_z^2 + U - \frac{8\lambda^2}{\pi} \leq 0$$

the charge excitation spectrum is gapless ($\Delta_c = 0$), and the fixed-point value of the parameter $K_c^*(l \rightarrow \infty)$ becomes unity.

Different from the charge mode, the anisotropy breaks the spin-SU(2) symmetry into a lower low U(1)-symmetry. This pushes the RG flows away from the separatrix. Depending
on the relation between the initial coupling constants, there are two different strong-coupling
sectors in the spin channel. For
\[
J_z > \max \left\{ \frac{8\lambda^2}{\pi} + U; \ J_\perp - \frac{8\lambda^2}{\pi} - U \right\}
\]
the spin excitation is massive ($\Delta_s > 0$). The RG flows scale to $-\infty$, and the phase field $\phi_s$
takes the average amplitude
\[
\langle \phi_s \rangle = 0.
\]
For
\[
J_z < \frac{\Delta_s}{8\lambda^2 + U} < J_\perp
\]
the spin channel is gapped ($\Delta_s > 0$). The RG flows scale to $+\infty$, and the phase field $\phi_s$ is
ordered with the average amplitude
\[
\langle \phi_s \rangle = \sqrt{\frac{\pi}{8K_s}}.
\]
In the other cases,
\[
J_z < \min \left\{ \frac{8\lambda^2}{\pi} + U; \ \frac{J_\perp}{2} \right\},
\]
the spin excitation is characterized by a massless mode ($\Delta_s = 0$). The low-energy behavior
of the model is governed by the fixed-point value of the Luttinger liquid parameter $K_s^*$. As a result, the charge-gap transition corresponds to
\[
J_\perp + J_z + \frac{U}{\pi} - \frac{8\lambda^2}{\pi} = 0
\]
whereas the spin-gap transition has two branches. One is
\[
J_z = J_\perp
\]
for $J_z \leq \left( \frac{8\lambda^2}{\pi} + U \right)$, and the other is
\[
J_z \leq \frac{8\lambda^2}{\pi} + U
\]
for $J_z > \frac{8\lambda^2}{\pi} + U$.

IV. QUANTUM PHASE DIAGRAM

We are in the position to acquire the weak-coupling quantum phase diagram of the
model Hamiltonian (4) at half filling.
IV-1. Order Parameters

In order to determine the dominated instabilities, we use a set of order parameters, which are defined in terms of bosonic field operators in the same way as in previous works [30, 44], describing the short wavelength fluctuations of the site-located (transverse and longitudinal) spin density,

\[ O_{SDW}^{\pm} \equiv (-1)^{j} \sum_{\alpha,\alpha'} c_{j,\alpha}^{\dagger} \sigma_{\alpha,\alpha'}^{\mp} c_{j,\alpha'} \]
\[ \sim \cos(\sqrt{2\pi K_c} \phi_c) \exp(\pm i\sqrt{2\pi / K_s} \theta_s), \] (39)

\[ O_{SDW}^{z} \equiv (-1)^{j} \sum_{\alpha,\alpha'} c_{j,\alpha}^{\dagger} \sigma_{\alpha,\alpha'}^{z} c_{j,\alpha'} \]
\[ \sim \cos(\sqrt{2\pi K_c} \phi_c) \sin(\sqrt{2\pi K_s} \phi_s), \] (40)

the site-located charge density,

\[ O_{CDW} = (-1)^{j} \sum_{\alpha} c_{j,\alpha}^{\dagger} c_{j,\alpha} \]
\[ \sim \sin(\sqrt{2\pi K_c} \phi_c) \cos(\sqrt{2\pi K_s} \phi_s), \] (41)

as well as the short wavelength fluctuations of the bond-located charge density,

\[ O_{BOW} = (-1)^{j} \sum_{\alpha} (c_{j,\alpha}^{\dagger} c_{j+1,\alpha} + h.c.) \]
\[ \sim \cos(\sqrt{2\pi K_c} \phi_c) \cos(\sqrt{2\pi K_s} \phi_s). \] (42)

In addition to these charge-gapped phases, there are superconducting order parameters:

\[ O_{TS}^{\pm} = \frac{1}{\sqrt{2}} (c_{j,\uparrow}^{\dagger} c_{j+1,\uparrow}^{\dagger} \pm c_{j,\downarrow}^{\dagger} c_{j+1,\downarrow}^{\dagger}) \]
\[ \sim \exp(i\sqrt{2\pi / K_c} \theta_c) \exp(i\sqrt{2\pi / K_s} \theta_s), \] (43)

\[ O_{TS}^{0} = \frac{1}{\sqrt{2}} \sum_{\alpha} c_{j,\alpha}^{\dagger} c_{j,\alpha}^{\dagger} \]
\[ \sim \exp(i\sqrt{2\pi / K_c} \theta_c) \sin(\sqrt{2\pi K_s} \phi_s), \] (44)

\[ O_{SS} = c_{j,\uparrow}^{\dagger} c_{j,\downarrow}^{\dagger} \]
\[ \sim \exp(i\sqrt{2\pi / K_c} \theta_c) \cos(\sqrt{2\pi K_s} \phi_s). \] (45)

IV-2. Quantum Phases

With the RG analysis of the low-energy excitation spectrum and the behavior of the order-parameters, we now make field-theoretical predictions for the quantum phase diagram of the system in the weak-coupling regime. We begin with the case for \( \lambda > \sqrt{\pi U/8} \) (see Fig. 2.).
FIG. 2: The schematic quantum phase diagram of the 1D deformed $t$-$J_\perp$-$J_z$ model.

Sector **A**: $J_z > 2(\frac{8\lambda^2}{\pi} + U) > J_\perp > J_z$.

The charge and spin excitation spectra are both massive ($\Delta_c, \Delta_s > 0$). The expectation values of the phase fields are obtained as

$$\langle \varphi_c \rangle = 0, \quad \langle \varphi_s \rangle = \sqrt{\frac{\pi}{8K_s}},$$  \hspace{1cm} (46)

and hence the order parameter $O_{\text{SDW}^z}$ takes the maximal value. Moreover, the correlation function shows

$$\langle O_{\text{SDW}^z}(x)O_{\text{SDW}^z}(x') \rangle \sim \text{constant}. \hspace{1cm} (47)$$

This corresponds to the existence of the longitudinal SDW$^z$ phase with a long-ranged order in the ground state.

Sector **B**: $\frac{8\lambda^2}{\pi} - J_\perp - U < J_z < \frac{8\lambda^2}{\pi} + U$.

The charge gap opens still while the spin gap closes ($\Delta_c > 0, \Delta_s = 0$). $\langle \varphi_c \rangle = 0$, but the phase field $\varphi_s$ is free. The U(1)-spin symmetry indicates that the low-energy property of the gapless degrees of freedom is controlled by the fixed-point value of the Luttinger liquid parameter $K_s^* > 1$.

$$\langle O_{\text{SDW}^z}(x)O_{\text{SDW}^z}(x') \rangle \sim |x - x'|^{-K_s^*},$$

$$\langle O_{\text{SDW}^\pm}(x)O_{\text{SDW}^\pm}(x') \rangle \sim |x - x'|^{-1/K_s^*}. \hspace{1cm} (48)$$

The SDW$^z$ correlation decays faster, so the transverse SDW$^\pm$ correlations dominate in the ground state.

Sector **C**: $\frac{8\lambda^2}{\pi} + U < J_z < J_\perp - \frac{8\lambda^2}{\pi} - U$. 


As in the sector A, the low-energy excitations are gapped in the charge and spin channels \((\Delta_c, \Delta_s > 0)\). Whereas, the expectation values of the phase fields are vanishing,
\[
\langle \phi_c \rangle = \langle \phi_s \rangle = 0.
\]
(49)

Obviously, the order parameter \(O_{\text{BOW}}\) takes the maximal value, and the correlation function exhibits
\[
\langle O_{\text{BOW}}(x)O_{\text{BOW}}(x') \rangle \sim \text{constant}.
\]
(50)

Therefore, the ground state is a LRO dimerized phase.

Sector D:
\[
\frac{8\lambda^2}{\pi} - \frac{J}{2} - U < J_\perp < J_z.
\]

In contrast to Sector B, the charge gap is closed but the spin is opened \((\Delta_c = 0, \Delta_s > 0)\). The charge field \(\phi_c\) is free, and the spin field is ordered with expectation value
\[
\langle \phi_s \rangle = \sqrt{\pi/8K_s}.
\]
(51)

As common in the half-filled case, the gapless charge excitation opens a possibility for the realization of a superconducting instability. The ordering with \(\langle \phi_s \rangle = \sqrt{\pi/8K_s}\) leads to the disappearance of the SS correlation, and the spin gap suppresses the TS\(^\pm\) correlations. The system is dominated in the TS\(^0\) phase.

Sector E:
\[
\frac{8\lambda^2}{\pi} - J_\perp - U < J_\perp < J_z.
\]

In this case the charge and spin channels are massless \((\Delta_c = \Delta_s = 0)\). The low-energy behavior of the system is controlled by the Luttinger liquid parameters \(K^*_c = 1\) and \(K^*_s > 1\). The phase fields \(\phi_c\) and \(\phi_s\) are free. The triplet superconducting correlations TS\(^\pm\) are favored in the ground state in suppression of the other phases.

We next consider the case for \(\lambda < \sqrt{\pi U/8}\). In the context of antiferromagnetic exchange, the charge gap opens everywhere, and hence the superconducting phases disappear. The ground state shows the insulating behavior.

**IV-3. Effects of the \(g_{cs}\) and \(\lambda\) terms**

Depending on \(\lambda \ll 1\) and the weak-coupling limit where the role of the charge-spin coupling may be safely omitted, we have obtained quantum phases of the model by the charge-spin separation theory. Now we examine how the charge-spin coupling part \(H_{cs}\) affects the phase diagram. With increasing interactions, the \(g_{cs}\) coupling becomes less irrelevant, at least in the intermediate- and strong-coupling regimes. Thus the perturbative RG technique is no longer justified. The present situation is distinct from the 1D repulsive Hubbard model, where the charge and spin degrees of freedom separate exactly in the whole coupling regime, indicating that the \(g_{cs}\) term is always irrelevant [45]. From the expression (25), the charge-spin coupling is influenced by the parameters \(\lambda\) and \(J_z\). One expects several
extremes ($\lambda = 0/1, J_z = 0/ + \infty$) to be the fixed-points of the 1D deformed $t$-$J_{\perp}$-$J_z$ model. We first exemplify the limiting case ($\lambda, J_z$)=($0, +\infty$) when the superconducting phases are absent. We can analyze this issue via the quasi-classical potentials [31, 39, 41],

\[ V_{\text{SDW}} = g_u - g_{\perp} + g_{cs}, \]
\[ V_{\text{BOW}} = g_u + g_{\perp} - g_{cs}. \]

It is easily found that, when $g_{cs} > g_{\perp}$, $V_{\text{BOW}} < V_{\text{SDW}}$. This hints that in the crossover regime, the ground state is in favor of the dimerized phase. Once the $g_{cs}$ coupling dominates over the $g_u$ and $g_{\perp}$ terms, the BOW phase disappears again in the strong-coupling regime. This situation is the same as the 1D extended Hubbard models [31, 39, 41].

We next consider the role of $\lambda$. In the case of $U, J \ll 1$ and $\lambda > \sqrt{8U/\pi}$, the triplet superconducting instabilities can survive. On the other hand, the introduction of $\lambda$ enhances the charge-spin coupling. This fact makes the parameter $\lambda$ more subtle. For larger $\lambda$, the weak-coupling argument is no longer valid. Here, we cannot accurately determine the quantum phases. However, it is believed that, when $\lambda = 1$, the BOW phase does not exist. This is because the BOW phase has no room to be accommodated in the case of the no-double-occupancy constraint.

V. DISCUSSION AND SUMMARY

At half filling, we studied the 1D deformed $t$-$J$ model with anisotropic antiferromagnetic exchange analytically. Compared to the $t$-$J$ model, the deformed version possesses the advantage that the induced two- and three-body interactions become softened when $\lambda$ is very small. This enables the deformed model to be tractable in the weak-coupling regime, where the bosonization and RG techniques can be safely applied. However, the introduction of a small $\lambda$ permits occurrence of double occupancy. In this case, the on-site $U$ term should play a significant role [see Eq. (4)]. We consider this point, which was neglected in the past work [24–26]. The existence of anisotropic exchange breaks the spin-SU(2) symmetry down into a U(1)-symmetry, leading to the removal of critical spin correlations. For $\lambda > \sqrt{\pi U/8}$, the weak-coupling phase diagram consists of five quantum phases, including the SDW$^{\pm}$, SDW$^{\pm}$, BOW, TS$^0$, and TS$^\pm$ instabilities. For $\lambda \leq \sqrt{\pi U/8}$, the triplet superconducting phases completely collapse and the insulating SDW$^z$, SDW$^{\pm}$, and BOW persist.

We briefly take into account roles of the $g_{cs}$ and $\lambda$ terms, which have an important effect on the quantum phase diagram for larger values. The result shows that the ground state phase diagram cannot evolve from $\lambda \ll 1$ into $\lambda = 1$. This is simply due to the fact that the Hilbert space of the $t$-$J$ model is not exactly equal to that of the deformed version. Moreover, It was proved that at half filling the 1D $t$-$J$ model is always in a SDW state [14]. Instead, besides the SDW phase, the TS and BOW phase can exist in the deformed version.

In summary, the 1D deformed $t$-$J_{\perp}$-$J_z$ model is not equivalent to the conventional $t$-$J$ model. The assumption of the adiabatical continuum cannot be applied to the $\lambda = 1$ case. It is expected that our study can provide a further understanding of low dimensional itinerant electron systems.
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References