Thermodynamics with Entropy Corrections in a Kaluza-Klein Universe

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In this paper, we focus our study on the generalized second law of thermodynamics with thermal equilibrium and non-equilibrium processes in a flat as well as in a closed Kaluza-Klein universe. We consider the universe to be composed of \( n \)-components of fluid interacting with each other in thermal equilibrium and two non-interacting components of fluid (such as dark energy and dark matter) not in thermal equilibrium. The validity of the generalized second law of thermodynamics is investigated in the scenario of power law and logarithmic corrections to the cosmological horizon entropy on the apparent as well as on the event horizon. Further, the behavior of this law in the quintessence and phantom dominated universe is analyzed. We find viable constraints for its validity in terms of the entropy corrected parameters \( \lambda \) and \( \mu \).

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I. INTRODUCTION

The phenomenon of the accelerated expansion of the universe is believed to be well understood in modified and extra dimensional theories of gravity. In this regard, many modified theories have been proposed such as \( f(R) \), \( f(G) \), \( f(R,G) \) \cite{1}, \( f(T) \) \cite{2}, Brans-Dicke \cite{3}, Horava-Lifshitz \cite{4}, etc. However, Kaluza-Klein (KK) theory \cite{5} is the pioneer among extra dimensional theories and has led to other theories like string theory \cite{6}, brane models \cite{7}, and supergravity \cite{8}. It is based on the idea of unification of gravity and electromagnetism which requires an extra dimension to explain this phenomenon.

Kaluza-Klein theory is classified into compact and non-compact versions on the basis of its fifth dimension role \cite{9}. It is argued that the fifth dimension introduces an effective length in the four dimensional projection that should be very small. This is known as compact KK theory. On the other hand, the fifth dimension behaves like a mass in non-compact KK theory, which originates from the well-known Campbell theorem. According to it, matter enters in the four dimensional theory through the five dimensional vacuum theory, and we cannot induce any matter into the five dimensional manifold by ourselves.

The concept of thermodynamics in a cosmological system comes from black hole physics. It was suggested \cite{10} that the temperature of the Hawking radiations emitting

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from black holes is proportional to their corresponding surface gravity on the event horizon. Jacobson [11] found a relation between thermodynamics and the Einstein field equations. He derived this relation on the basis of an entropy-horizon area proportionality relation along with the first law of thermodynamics (also called the Clausius relation) \( dQ = TdS \), where \( dQ \), \( T \), and \( dS \) indicate the exchange in energy, temperature, and entropy change for a given system. It was shown that the field equations for any spherically symmetric spacetime can be expressed as \( TdS = dE + PdV \) (where \( E \), \( P \), and \( V \) represent the internal energy, pressure and volume of the spherical system) for any horizon [12].

The generalized second law of thermodynamics (GSLT) has been studied extensively in the scenario of the expanding behavior of the universe. This expansion is due to the presence of one of the major constituents of fluid named as dark energy (DE). The GSLT states that the entropy of the matter inside the horizon plus the entropy of the horizon remains positive and increases with the passage of time [13]. In order to discuss the GSLT, the horizon entropy of the universe can be taken as one quarter of its horizon area [14] or in a power law corrected [15] or logarithmic corrected form [16]. Many people have explored the validity of the GSLT of different systems, including the interaction of two components of fluid like DE and DM [17] and the interaction of three components of fluid [18] in the FRW universe by using a simple horizon entropy of the universe. Also, Sharif and Zubair [19] have examined the validity of the first and second law of thermodynamics on the apparent horizon in \( f(R, T) \) gravity.

Sharif and his collaborators [20–22] have discussed the GSLT in the interacting scenario of modified holographic DE in a flat and a non-flat KK universe with the help of a simple horizon entropy of the universe. The logarithmic and power law corrections to the usual entropy-area relation of the expanding universe may be useful in exploring the GSLT. Some people [23, 24] have investigated the GSLT of a system containing an accretion process of a phantom fluid onto the Schwarzschild black hole and logarithmic as well as power law corrected entropies on the apparent and event horizons in the FRW universe. Recently, some people have discussed the validity conditions of the GSLT for logarithmic [25] and power law entropy [26] corrected versions in the flat FRW universe. They have considered a system of interacting \( n \)-components of fluid on the apparent and event horizons by taking into account the thermal equilibrium and non-equilibrium conditions.

In this work, we follow the procedure of [25, 26] and discuss the GSLT in a flat and a closed KK universe. We consider both entropy corrected versions of the horizon entropy of the universe on the apparent and event horizons. The outline of this paper is as follows. Section II contains the general formalism of the GSLT with thermal equilibrium. In Sections III and IV, we discuss the GSLT in a flat and a closed KK universe, respectively. Section V is devoted to the study of the GSLT in a flat and a closed KK universe with thermal non-equilibrium. In the last section, we summarize our results.
II. KALUZA-KLEIN UNIVERSE AND BASIC EQUATIONS OF THE GSLT

In this section, we formulate the basic results of the GSLT for a system containing an interaction of $n$-components of fluid with thermal equilibrium and corrected entropies of the cosmological horizon, e.g., power law and logarithmic forms of entropy. For this purpose, we consider the KK universe [27],

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) + (1 - kr^2)d\psi^2 \right],$$

where $a(t)$ denotes the cosmic scale factor which measures the expansion of the universe and $k = 0, 1, -1$ correspond respectively to a flat, closed, and open KK universe. The corresponding field equations are

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{6} \rho,$$  \hspace{1cm} (2)

$$\dot{H} + 2H^2 + \frac{k}{a^2} = -\frac{8\pi}{3} p.$$  \hspace{1cm} (3)

Here $\rho$ and $p$ represent the total energy density and pressure of the fluid in the universe.

Now, consider the $n$-component fluid for the interacting scenario to exchange energy. The DE, DM, and radiation have a major contribution in this fluid. We assume here that all components flow with the same four-velocity, which implies that $\rho = \sum_{i=1}^{n} \rho_i$ and $p = \sum_{i=1}^{n} p_i$ represent the sum of the energy density and pressure of the $n$-components of the fluid. The equation of continuity in the interacting scenario takes the form

$$\dot{\rho}_i + 4H(\rho_i + p_i) = \Upsilon_i, \quad i = 1, 2, 3, \ldots, n,$$  \hspace{1cm} (4)

where $\Upsilon_i$ shows the interacting parameter such that $\sum_{i=1}^{n} \Upsilon_i = 0$ whose different specific forms are available in the literature [28]. This term makes possible energy exchange between different constituents of the perfect fluid and is helpful for solving the coincidence problem. The evolution history of the universe from early deceleration to late time acceleration was also investigated through interacting fluids [29]. Moreover, the transition of the equation of state parameter of different DE models from quintessence towards phantom eras may arise in the context of interacting fluids [30]. Observational data favors these types of models [31].

To check the validity of the GSLT, we start from the Gibb's relation [32] for each component of fluid, which has the following form:

$$T_i dS_i = d(\rho_i V) + p_i dV,$$  \hspace{1cm} (5)

here, $T_i$, $S_i$, and $V = \frac{\pi^2 R_h^4}{2}$ represent the temperature, entropy of the $i$th component of fluid, and volume of a spherical system in four dimensions, respectively. $R_h$ describes the radius of the horizons (event and apparent) of the universe whose two choices are extensively used in the debate of accelerated expansion of the universe. These are helpful in describing
the entropy along the boundary of the universe. The apparent horizon is the first one which is defined in a non-flat KK universe as \[33\]

\[
R_a = \frac{1}{H\sqrt{1 + \Omega_k}}. \tag{6}
\]

The event horizon is another form of horizon, which is defined as \[34\]

\[
R_e = a(t) \int_{a}^{\infty} \frac{d\tilde{a}}{H\tilde{a}^2} = a(t) \int_{t}^{\infty} \frac{d\tilde{t}}{a(\tilde{t})}. \tag{7}
\]

This has attracted much attention after the description of holographic DE.

In this work, we use these two horizons for the sake of generality. Differentiation of the above equation with respect to time yields

\[
\dot{R}_e = HR_e - 1. \tag{8}
\]

Notice that \(\dot{H} < 0\) and \(\dot{R}_e > 0\) (\(HR_e > 1\)) for the quintessence dominated era and vice versa in the phantom dominated universe \[35\]. Making use of the volume and differentiation of (5) with respect to time, we obtain the rate of change of the interacting fluid entropy,

\[
\dot{S}_i = \frac{\pi^2 R_h^4}{2} \frac{\Upsilon_i}{T_i} + 2\pi^2 R_h^3 (\dot{R}_h - HR_h) \frac{\rho_i + p_i}{T_i}. \tag{9}
\]

Consequently, the rate of change of the sum of \(n\)-components of the interacting fluid becomes

\[
\dot{S} = \sum_{i=1}^{n} \dot{S}_i = \frac{\pi^2 R_h^4}{2} \sum_{i=1}^{n} \frac{\Upsilon_i}{T_i} + 2\pi^2 R_h^3 (\dot{R}_h - HR_h) \sum_{i=1}^{n} \frac{\rho_i + p_i}{T_i}. \tag{10}
\]

We proceed with the thermal equilibrium condition \((T_i = T_h = (2\pi R_h)^{-1})\), which states that the temperature of the \(n\)-components of the fluid is the same as the temperature of the horizon.

**II-1. Entropy Corrections**

The horizon entropy of the universe plays an important role in the analysis of the GSLT of the expanding universe. For this purpose, usually, the Bekenstein entropy-area relation \((S \propto A)\), \(S_h = \frac{4}{3} A\), is used, where \(A = 4\pi R_h^2\) appears as an area of the spherical system. However, the corrections to this relation occur in the entanglement of quantum fields \[36\]. It has been examined that the origin of the usual entropy may exist in the entanglement of quantum fields between the inside and outside of the horizon. Indeed, the modes of gravitational fluctuations lie in the background of a black hole acting as scalar fields, and its entanglement entropy can be calculated by tracing degrees of freedom inside the horizon. The entanglement entropy is defined as *a measure of the information loss due to the spatial separation between the degrees of freedom inside and outside the horizon*. It was shown \[36\] that the Bekenstein entropy-area relation is valid only for the field in its
ground state. While in case of a superposition of the ground and excited states of a field, a correction term in the power law form of area should be added. In this respect, the correction is found in the power law form as \[15, 36\]

\[ S_h = \frac{A}{4} \left[ 1 - B_\lambda A^{1 - \frac{1}{2}} \right], \tag{11} \]

where \( B_\lambda = \frac{\lambda (4\pi)\frac{3}{2} - 1}{(4 - \lambda) r_c^{2 - \lambda}} \), \( \lambda \) is a constant and \( r_c \) is the crossover scale.

The other correction of the Bekenstein entropy-area relation is found in the logarithmic form in loop quantum gravity. In this context, the entropy-area relation can be expanded in an infinite series given by

\[ S_h = \frac{A}{4\hbar} + \tilde{\alpha} \ln \left( \frac{A}{4\hbar} \right) - \tilde{\alpha}_1 \left( \frac{4\hbar}{A} \right) - \tilde{\alpha}_2 \left( \frac{16\hbar^2}{A^2} \right) - ... \]

\[ = S_0 + \tilde{\alpha} \ln S_0 - \sum_{j=1}^{\infty} \frac{\tilde{\alpha}_j}{S_0^j}, \]

here \( \tilde{\alpha}_j \) are finite constants and \( \hbar \) is the Planck constant. Also, \( S_0 \) indicates the classical black hole entropy and the other terms are known as quantum corrections. It is noted that only the first order correction is suitable for cosmological analysis, while the higher order can be neglected due to the small value of \( \hbar \). The logarithmic entropy correction in the scenario of thermal fluctuations around equilibrium has also been found \[37\]. In the following, we consider only the first order logarithmic correction to the entropy \[16\]:

\[ S_h = \frac{A}{4} + \mu \ln \left( \frac{A}{4} \right) + \nu, \tag{12} \]

where \( \hbar \) is assumed to be unity and \( \mu, \nu \) are constants.

It is noted that entropy corrections from quantum effects appear in the area, which modifies the Bekenstein entropy-area relationship and the result for the entropy of the horizon. However, the entropy of all the \( n \)-components of the fluid in the horizon is described by the Gibb’s relation, which does not involve the area but rather the radius of the horizon. Thus we may take the corrected horizon entropy from quantum effects and the Gibb’s relation from the classical fluid approach to discuss the GSLT. Cai et al. \[38\] suggested that the logarithmic term possesses the property of unification of the early inflation as well as present acceleration. They also found the value of the coefficient of the logarithmic term \( \mu \) through current cosmological constraints of the order of \( 10^{16} \), which may raise the fine tuning problem in the entropy corrected model. Thus it would be interesting to find the appropriate value of \( \mu \) through different observational schemes and make the logarithmic correction more reliable in the current cosmological scenario. The entropy-area relation is also modified as \( S \propto f'(R)A \) in \( f(R) \) gravity \[39\].
III. GSLT IN THE FLAT KALUZA-KLEIN UNIVERSE

Here we explore the GSLT on the apparent and event horizons in the flat KK universe ($k = 0$) successively. Also, we take an example of the expansion of the Hubble parameter with small perturbation on the apparent horizon and a choice of a pole-like scale factor in the event horizon. For the power law corrected entropy, the variation of (11) with respect to time is given by

$$\dot{S}_h = \frac{3\pi^2}{2} R_h^2 \dot{R}_h \left( 1 - \frac{\lambda}{r_c^{2-\lambda}} \left( \frac{\pi R_h^3}{2} \right)^{1-\frac{\lambda}{2}} \right).$$ \hspace{1cm} (13)

In this correction, the general form of the GSLT can be formulated with the help of Eqs. (2), (10), (13), and the thermal equilibrium condition

$$\dot{S}_{tot} = \frac{3\pi^2}{2} R_h^2 \dot{R}_h \left[ \dot{R}_h^2 (HR_h - \dot{R}_h) + \dot{R}_h \left( 1 - \frac{\lambda}{r_c^{2-\lambda}} \left( \frac{\pi R_h^3}{2} \right)^{1-\frac{\lambda}{2}} \right) \right].$$ \hspace{1cm} (14)

The time variation of the horizon entropy of the universe with logarithmic correction is

$$\dot{S}_h = \frac{3\pi^2}{2} R_h^2 \dot{R}_h \left( 1 + \frac{2\mu}{\pi^2 R_h^3} \right).$$ \hspace{1cm} (15)

The corresponding general form of the GSLT expression turns out to be

$$\dot{S}_{tot} = \frac{3\pi^2}{2} R_h^2 \dot{R}_h \left[ \dot{R}_h^2 (HR_h - \dot{R}_h) + \dot{R}_h \left( 1 + \frac{2\mu}{\pi^2 R_h^3} \right) \right].$$ \hspace{1cm} (16)

Using Eq. (6) in (14), the GSLT for the power law entropy corrected on the apparent horizon yields

$$\dot{S}_{tot} = \frac{3\pi^2 H}{2H^4} \left[ \dot{H} + \frac{\lambda}{r_c^{2-\lambda}} \left( \frac{\pi}{2H^2} \right)^{1-\frac{\lambda}{2}} \right].$$ \hspace{1cm} (17)

This shows that the GSLT is valid for all times when $\lambda = 0$.

We can also analyze this result by using Taylor’s expansion of the Hubble parameter about $t = t_0$ (present time), which is used for small perturbations around the de Sitter space [25]. It is given by

$$H = H_0 + H_0^2 \delta + O(\delta^2), \quad \delta = \frac{\dot{H}}{H^2}, \quad |\delta| \ll 1.$$ \hspace{1cm} (18)

Using this value in Eq. (17), we conclude that the GSLT holds when $\lambda \leq 0$ or $\lambda \geq 0$ with $r_c < \left( -\frac{1}{2} \right)^{\frac{1}{2-\lambda}} \sqrt{\frac{2}{\pi H^2}}$ in the quintessence case (having $\dot{H} < 0$), while in the phantom...
dominated case (having $\dot{H} > 0$), $\lambda \geq 0$ or $\leq 0$ with $r_c > \left(\frac{1}{2}\right)^{\frac{1}{2-\lambda}}\sqrt{\frac{\dot{H}}{2\pi^2}}$. The GSLT of the logarithmic correction to entropy on the apparent horizon leads to

$$\dot{S}_{\text{tot}} = \frac{3\pi^2\dot{H}}{2H^4} \left[ \frac{\dot{H}}{H^2} - \frac{2\mu H^3}{\pi^2} \right].$$

(19)

In view of Eq. (18), we find that it holds for $\mu \geq 0$ or $\mu \leq 0$ with $\delta < \frac{2\mu H^3}{\pi^2}$ in the quintessence dominated case, while for $\mu \leq 0$ or $\mu \geq 0$ with $\delta > \frac{2\mu H^3}{\pi^2}$ in the phantom dominated case.

The GSLT for power law corrected entropy on the event horizon becomes

$$\dot{S}_{\text{tot}} = \frac{3\pi^2 R_c^2}{2} \left[ \dot{H} R_c^2 + \dot{R}_c \left( 1 - \frac{\lambda}{r_c^2-\lambda} \left( \frac{\pi R_c^3}{2} \right)^{1-\frac{1}{2}} \right) \right].$$

(20)

In the quintessence region, it holds for $\lambda \leq 0$ with

$$\dot{R}_c \geq -\dot{H} R_c^2 \left( 1 - \frac{\lambda}{r_c^2-\lambda} \left( \frac{\pi R_c^3}{2} \right)^{1-\frac{1}{2}} \right)^{-1}.$$

However, in the phantom region, the GSLT (20) holds for $\lambda \leq 0$. In a similar way, we obtain the GSLT for logarithmic type entropy

$$\dot{S}_{\text{tot}} = \frac{3\pi^2 R_c^2}{2} \left[ \dot{H} R_c^2 + \dot{R}_c \left( 1 + \frac{2\mu}{\pi^2 R_c^3} \right) \right].$$

(21)

For its validity, $\mu \geq 0$ with $\dot{R}_c \geq -\frac{\pi^2 H R_c^3}{\pi^2 R_c^2 + 2\mu}$ in the quintessence case and $\mu \geq 0$ or $\mu \leq 0$ with $-\pi^2 R_c^3 < 2\mu$ in the phantom case.

In order to illustrate the results (20) and (21), we take a pole-like scale factor in the form [25]

$$a(t) = a_0(t_s - t)^{-p}, \quad a_0 > 0, \quad p > 0, \quad t_s > t.$$  

(22)

This corresponds to the phantom dominated universe, where super accelerated expansion with big rip singularity exists at finite time $t_s$. At this stage, the Hubble parameter goes towards a very large value. In terms of this scale factor, we have

$$H = \frac{p}{t_s - t}, \quad R_e = \frac{t_s - t}{p + 1}, \quad \dot{H} = \frac{p}{(t_s - t)^2}.$$  

(23)

These quantities yield the GSLT of power law entropy corrected in the form

$$\dot{S}_{\text{tot}} = \frac{3\pi^2 (t_s - t)^2}{2(p + 1)^3} \left[ -\frac{1}{p + 1} + \frac{\lambda}{r_c^2-\lambda} \left( \frac{\pi (t_s - t)^3}{2(p + 1)^3} \right)^{1-\frac{1}{2}} \right].$$

(24)
which holds for
\[ t_s - t \geq (p + 1) \left[ \frac{2}{\pi} (p + 1)^{\frac{\lambda}{p} - 1} \right]^{\frac{3}{p}}. \] (25)

Using Eqs. (22) and (25), we have the following limit on the scale factor:
\[ a(t) \leq a_0 (p + 1)^{-p} \left[ \frac{2}{\pi} (p + 1)^{\frac{\lambda}{p} - 1} \right]^{-\frac{2}{3}}. \] (26)

It is the upper bound of the scale factor below which the GSLT is satisfied for the power law entropy corrected case. In the case of logarithmic corrected entropy, the GSLT takes the form
\[ \dot{S}_{tot} = \frac{3\pi^2 (t_s - t)^2}{2(p + 1)^3} \left[ -\frac{1}{p + 1} - \frac{2\mu (p + 1)^3}{\pi^2 (t_s - t)^3} \right]. \] (27)

For \( \dot{S}_{tot} \geq 0 \), it follows that
\[ t_s - t \leq \left[ -\frac{2\mu (p + 1)^4}{\pi^2} \right]^{\frac{3}{4}}, \] (28)

which is true only for \( \mu \leq 0 \) and also gives the limit on the scale factor
\[ a(t) \geq a_0 \left[ -\frac{2\mu (p + 1)^4}{\pi^2} \right]^{-\frac{2}{3}} \] (29)

for the validity of the GSLT.

In order to give an illustration of the validity through observational constraints on the cosmological parameters, we restrict this work to two main components of the universe, i.e., DE and cold dark matter (CDM) for analyzing the behavior of the GSLT. In the presence of DE and CDM components with a flat KK universe, Eq. (2) in terms of fractional energy densities can be written as
\[ 1 = \Omega_m + \Omega_\Lambda, \quad \Omega_m = \frac{\rho_m}{6m_p^2 H^2}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{6m_p^2 H^2}. \] (30)

where \( m_p^2 = (8\pi G)^{-1} \). By differentiating Eq. (2) and using Eq. (4) for DE and CDM only, we obtain
\[ \frac{\dot{H}}{H^2} = -\frac{1}{2} (1 + \omega_\Lambda \Omega_\Lambda). \] (31)

In this scenario, the GSLT expressions corresponding to power law and logarithmic entropy corrected with the apparent as well as event horizon are summarized as follows:
For power law entropy corrected on the apparent horizon, inserting Eq. (31) in (17), we get

\[ \dot{S}_{\text{tot}} = \frac{3\pi^2}{4H^2}(1 + \omega\Omega) \left[ \frac{1}{2}(1 + \omega\Omega) - \frac{\lambda}{r_c^{2-\lambda}} \left( \frac{\pi}{2H^3} \right)^{1-\frac{\lambda}{2}} \right]. \]  

(32)

Here, we introduce the observational values of different cosmological parameters at the present epoch, i.e., \( H_0 = 71 \), \( \omega\Lambda_0 = -1.084 \), \( \Omega\Lambda_0 = 69.2 \) [40]. In view of these values, the above expression becomes

\[ \dot{S}_{\text{tot}} = 4.0178 + 0.1088(4.39 \times 10^{-6})^{1-\frac{\lambda}{2}} \frac{\lambda}{r_c}. \]

This clearly shows that the GSLT holds at the present epoch.

For logarithmic entropy corrected on the apparent horizon, the expression (19) takes the form

\[ \dot{S}_{\text{tot}} = \frac{3\pi^2}{4H^2}(1 + \omega\Omega) \left[ \frac{1}{2}(1 + \omega\Omega) + \frac{2\mu H^3}{\pi^2} \right]. \]  

(33)

In view of observational values of cosmological parameters, the above expression takes the form

\[ \dot{S}_{\text{tot}} = 4.0178 - 7862.33\mu. \]

In this case, the GSLT holds only for \( \mu \leq 0 \).

For power law entropy corrected on the event horizon, the expression (24) in terms of \( H \) becomes

\[ \dot{S}_{\text{tot}} = \frac{3\pi^2 p^2}{2(p + 1)^3 H^2} \left[ -\frac{1}{p + 1} + \frac{\lambda}{r_c^{2-\lambda}} \left( \frac{\pi p^3}{2(p + 1)^3 H^3} \right)^{1-\frac{\lambda}{2}} \right], \]  

(34)

As we can see from above expression that the GSLT holds for \( H_0 = 71 \) [40] and \( \lambda \geq 2 \) at the present epoch.

For logarithmic entropy corrected on the event horizon, the expression (27) in terms of \( H \) turns out to be

\[ \dot{S}_{\text{tot}} = \frac{3\pi^2 p^2}{2(p + 1)^3 H^2} \left[ -\frac{1}{p + 1} - \frac{2\mu(p + 1)^3 H^3}{\pi^2 p^3} \right]. \]  

(35)

This expression shows that the GSLT holds for \( H_0 = 71 \) [40] and \( \mu < 0 \) at the present epoch.
IV. GSLT IN A CLOSED KALUZA-KLEIN UNIVERSE

In this section, we check the validity of the GSLT of interacting \( n \)-components of fluid with entropy corrected versions in the closed KK universe on the apparent as well as event horizon. The time variation of entropy of the whole fluid in a closed KK universe can be calculated by using Eqs. (2) and (10) as

\[
\dot{S} = \frac{3\pi^2}{2} H^2 R_h^4 (H R_h - \dot{R}_h) \left( \frac{\dot{H}}{H^2} - \Omega_k \right). \tag{36}
\]

For the power law corrected entropy, the GSLT relation is obtained by combining Eqs. (13) and (36) as

\[
\dot{S}_{\text{tot}} = \frac{3\pi^2}{2} H^2 R_h^2 (H R_h - \dot{R}_h) \left( \frac{\dot{H}}{H^2} - \Omega_k \right) + \dot{R}_h \left( 1 - \frac{\lambda}{r_c^{2-\lambda}} \left( \frac{\pi R_h^3}{2} \right)^{1-\frac{2}{\lambda}} \right). \tag{37}
\]

For the logarithmic correction, the GSLT leads to

\[
\dot{S}_{\text{tot}} = \frac{3\pi^2}{2} H^2 R_h^2 (H R_h - \dot{R}_h) \left( \frac{\dot{H}}{H^2} - \Omega_k \right) + \dot{R}_h \left( R_h + \frac{2\mu}{\pi^2 R_h^3} \right). \tag{38}
\]

Now, we evaluate the results of the GSLT corresponding to power law and logarithmic forms of the entropies on the apparent and event horizons, respectively. Inserting the value of the apparent horizon from Eq. (6) in (37), it follows that

\[
\dot{S}_{\text{tot}} = \frac{3\pi^2}{2H^2 (1 + \Omega_k)^{\frac{3}{2}}} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ \frac{\lambda}{r_c^{2-\lambda}} \left( \frac{\pi}{2H^3 (1 + \Omega_k)^{\frac{3}{2}}} \right)^{1-\frac{2}{\lambda}} \right] + \frac{1}{1 + \Omega_k} \left( \frac{\dot{H}}{H^2} - \Omega_k \right). \tag{39}
\]

The GSLT of power law corrected entropy on the apparent horizon holds when \( \lambda \leq 0 \) or \( \lambda \geq 0 \) with

\[
\frac{\dot{H}}{H^2} < -\frac{\lambda}{r_c^{2-\lambda}} \left( \frac{\pi}{2H^3} \right)^{1-\frac{2}{\lambda}} \left( 1 + \Omega_k \right)^{\frac{3}{2}(3\lambda-2)} + \Omega_k \tag{40}
\]

in the quintessence region \( \frac{\dot{H}}{H^2} < \Omega_k \). In the phantom case, the GSLT is respected for \( \frac{\dot{H}}{H^2} > \Omega_k \) when \( \lambda \geq 0 \) or \( \lambda \leq 0 \) with

\[
\frac{\dot{H}}{H^2} > -\frac{\lambda}{r_c^{2-\lambda}} \left( \frac{\pi}{2H^3} \right)^{1-\frac{2}{\lambda}} \left( 1 + \Omega_k \right)^{\frac{3}{2}(3\lambda-2)} + \Omega_k. \tag{41}
\]
Also, the GSLT holds (i.e., $\dot{S}_{\text{tot}} \geq 0$ of Eq. (39)) for $\frac{\dot{H}}{H^2} < \Omega_k$ when $\lambda \leq 0$ or $\lambda \geq 0$ along with the inequality (40). The GSLT of logarithmic corrected entropy on the apparent horizon turns out to be

$$\dot{S}_{\text{tot}} = \frac{3\pi^2}{2H^2(1 + \Omega_k)^4} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \sqrt{1 + \Omega_k} \left( \frac{2\mu H^3}{\pi^2} \right) \right], \quad (42)$$

which remains valid for $\mu \geq 0$ in the quintessence case. In the phantom dominated era, it holds for $\frac{\dot{H}}{H^2} > \Omega_k$ along with $\mu \leq 0$ or with the combination ($\mu \geq 0$, $\frac{\dot{H}}{H^2} > \frac{2\mu}{\pi^2 H^3 \sqrt{1 + \Omega_k}} + \Omega_k$).

Also, the GSLT holds in the above expression for $\frac{\dot{H}}{H^2} < \Omega_k$ along with $\mu \geq 0$ or with the combination ($\mu \leq 0$, $\frac{\dot{H}}{H^2} < \frac{2\mu}{\pi^2 H^3 \sqrt{1 + \Omega_k}} + \Omega_k$).

The GSLT of power law entropy on the event horizon is found through Eqs. (7) and (37):

$$\dot{S}_{\text{tot}} = 3\pi^2 \frac{R^2}{2} \left[ H^2 \dot{R}^e \left( \frac{\dot{H}}{H^2} - \Omega_k \right) + \dot{R}^e \left( 1 - \frac{\lambda \rho}{r^2 - \lambda} \left( \frac{\pi R^3}{2} \right)^{1 - \frac{1}{p}} \right) \right]. \quad (43)$$

In the quintessence region, it holds for $\lambda \leq 0$ with

$$\dot{R}^e \geq - \frac{H^2 \dot{R}^e \left( \frac{\dot{H}}{H^2} - \Omega_k \right)}{1 - \frac{\lambda \rho}{r^2 - \lambda} \left( \frac{\pi R^3}{2} \right)^{1 - \frac{1}{p}}}. \quad (44)$$

Also, in the phantom case, the GSLT (43) remains valid either for $\frac{\dot{H}}{H^2} > \Omega_k$ with $\lambda \leq 0$ or $\frac{\dot{H}}{H^2} < \Omega_k$ along with inequality (44). In case of logarithmic corrected entropy, the GSLT on the event horizon leads to

$$\dot{S}_{\text{tot}} = 3\pi^2 \frac{R^2}{2} \left[ H^2 \dot{R}^e \left( \frac{\dot{H}}{H^2} - \Omega_k \right) + \dot{R}^e \left( 1 + \frac{2\mu}{\pi^2 R^3} \right) \right]. \quad (45)$$

One can observe from the above expression that the GSLT holds, in the quintessence region, for $\mu \geq 0$ along with following inequality:

$$\dot{R}^e \geq \pi^2 H^2 R^5 \frac{\left( - \frac{\dot{H}}{H^2} + \Omega_k \right)}{\pi^2 R^3 + 2\mu}. \quad (46)$$

The GSLT (45) remains valid in the phantom case for $\frac{\dot{H}}{H^2} > \Omega_k$ along with $\mu \geq 0$ or $\mu \leq 0$ with $-\pi^2 R^3 < 2\mu$. Also, it remains valid for $\frac{\dot{H}}{H^2} < \Omega_k$ along with $\mu \geq 0$ and the inequality (46). Now we investigate the results of the GSLT on the event horizon for power law and logarithmic corrected entropies in terms of a phantom like scale factor. Using Eqs. (22) in (43), gives the following form of the GSLT for power law corrected entropy:

$$\dot{S}_{\text{tot}} = \frac{3\pi^2 (t_s - t)^2}{2(p + 1)^3} \left[ -1 + \frac{\lambda \rho}{r^2 - \lambda} \left( \frac{\pi(t_s - t)^3}{2(p + 1)^3} \right)^{1 - \frac{1}{p}} + \frac{p}{p + 1}(1 - \Omega_k p) \right], \quad (47)$$
which implies that
\[ t_s - t_0 \leq (p + 1) \left[ \frac{2r_c^2}{\pi} \left( \frac{p + 1}{\lambda} \right)^{\frac{2}{2-p}} \left( \frac{1}{p + 1} - \frac{p}{(p + 1)^2 (1 - \Omega_k p)^{2/2-p}} \right) \right]^{-\frac{1}{3}}. \] (48)

In terms of the scale factor, the GSLT of the logarithmic entropy leads to
\[ \dot{S}_{tot} = \frac{3\pi^2(t_s - t)^2}{2(p + 1)^3} \left[ \frac{p}{p + 1} (1 - \Omega_k p) - \left( 1 + \frac{2\mu (p + 1)^3}{\pi^2 (t_s - t)^3} \right) \right], \] (49)
which is true for
\[ t_s - t_0 \geq (p + 1) \left( \frac{2\mu}{\pi^2} \right)^\frac{1}{3} \left[ -1 + \frac{p}{p + 1} (1 - \Omega_k p) \right]^{-\frac{1}{3}}. \] (50)

In the presence of DE and CDM components with a closed KK universe, Eq. (2) in terms of fractional energy densities can be written as
\[ 1 + \Omega_k = \Omega_m + \Omega_\Lambda, \quad \Omega_k = \frac{k}{a^2 H^2}, \quad \Omega_m = \frac{\rho_m}{6m_p^2 H^2}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{6m_p^2 H^2}. \] (51)

The corresponding rate of change of the Hubble parameter takes the form
\[ \frac{\dot{H}}{H^2} = -\frac{1}{2} (1 - \Omega_k + \omega_\Lambda \Omega_\Lambda). \] (52)

In this scenario, the GSLT expressions corresponding to power law and logarithmic entropy corrected with apparent as well as event horizon are summarized as follows:

- For power law entropy corrected on the apparent horizon, inserting Eq. (52) in (39), we get
\[ \dot{S}_{tot} = -\frac{3\pi^2}{4H^2 (1 + \Omega_k)^{\frac{2}{2-p}}} (1 + \Omega_k + \omega_\Lambda \Omega_\Lambda) \left[ \frac{\lambda}{r_c^2 - \lambda} \left( \frac{\pi}{2H^3 (1 + \Omega_k)^{\frac{2}{2-p}}} \right) \right]^{1 - \frac{1}{2}} - \frac{0.5}{1 + \Omega_k} \left( 1 + \Omega_k + \omega_\Lambda \Omega_\Lambda \right). \] (53)

At the present epoch, the observational values are \( H_0 = 71, \quad \omega_{\Lambda 0} = -1.084, \quad \Omega_{\Lambda 0} = 69.2, \quad \Omega_{k 0} = 0.008 \) [40]. In view of these values, the above expression becomes
\[ \dot{S}_{tot} = 3.9065 + 0.1064 (4.3344 \times 10^{-6})^{1 - \frac{1}{2}} \lambda \frac{\lambda}{r_c}. \]

This clearly shows that the GSLT holds at the present epoch.
For logarithmic entropy corrected on the apparent horizon, the expression (42) takes the form
\[
\dot{S}_{\text{tot}} = \frac{3\pi^2}{4H^2(1 + \Omega_k)^3} (1 + \Omega_k + \omega_\Lambda \Omega_\Lambda) \left[ \frac{1}{2} (1 + \Omega_k + \omega_\Lambda \Omega_\Lambda) \sqrt{1 + \Omega_k + \frac{2\mu H^3}{\pi^2}} \right]. \tag{54}
\]
In view of the observational constraints, the above expression turns out to be
\[
\dot{S}_{\text{tot}} = 3.9064 - 7634.27\mu.
\]
Hence, the GSLT valid for \( \mu \leq 0 \).

For power law entropy corrected on the event horizon, the expression (47) in terms of \( H \) takes the form
\[
\dot{S}_{\text{tot}} = \frac{3\pi^2 p^2}{2H^2(p + 1)^3} \left[ -1 + \frac{\lambda}{r^2_c - \lambda} \left( \frac{p}{2(p + 1)H^3} \right)^{1/2} \frac{p}{p + 1} (1 - p\Omega_k) \right]. \tag{55}
\]
It can be seen that the GSLT holds for \( H_0 = 71, \ \Omega_{k0} = 0.008 [40], \) and \( \lambda \geq 2 \) at the present epoch.

For logarithmic entropy corrected on the event horizon, the expression (49) in terms of \( H \) turns out to be
\[
\dot{S}_{\text{tot}} = \frac{3\pi^2 p^2}{2(p + 1)^3H^2} \left[ \frac{p}{p + 1} (1 - p\Omega_k) - 1 - \frac{2\mu(p + 1)^3H^3}{\pi^2 p^3} \right]. \tag{56}
\]
As we can see from the above expression that the GSLT holds for \( H_0 = 71, \ \Omega_{k0} = 0.008 [40] \) and \( \mu < 0 \) at the present epoch.

V. GSLT WITH THERMAL NON-EQUILIBRIUM

Here we explore the GSLT for the non-equilibrium case. We consider a simple system which includes two components of fluid such as DM and DE in the universe. These components are non-interacting, denoted by \( \rho_m \) and \( \rho_\Lambda \) and satisfy the following set of equations:
\[
\dot{\rho}_m + 4H(1 + \omega_m)\rho_m = 0, \quad \dot{\rho}_\Lambda + 4H(1 + \omega_\Lambda)\rho_\Lambda = 0. \tag{57}
\]
We use the equations of state \( p_m = \rho_m \omega_m \) and \( p_\Lambda = \rho_\Lambda \omega_\Lambda \) for DM and DE, respectively, while \( \omega_m \) and \( \omega_\Lambda \) represent the corresponding parameters.
First we discuss the GSLT for the flat KK universe. We can calculate the following rate of change of fluid having two components with the help of Eqs. (2) (for the flat KK case), (9), and (57):

\[
\dot{S} = \pi R_h^3 (\dot{R}_h - HR_h) \left[ 2\pi (1 + \omega_m) \rho_m \left( \frac{1}{T_m} - \frac{3\dot{H}}{4T\Lambda} \right) \right],
\]

where \(T_m\) and \(T\Lambda\) describe the temperature of the DM and DE components of fluid, respectively. Using \(\rho_m = \rho_0 a^{-4(1+\omega_m)}\) (which can be obtained from (57)) and \(T\Lambda = \frac{1}{2\pi R_h}\) in the above equation, it follows that

\[
\dot{S} = \pi R_h^3 (\dot{R}_h - HR_h) \left[ 2\pi (1 + \omega_m) \rho_0 a^{-4(1+\omega_m)} \left( \frac{1}{T_m} - \frac{2\pi R_h}{2\pi} \right) - \frac{3\pi H R_h}{2H} \right],
\]

where \(\rho_0\) is the value of the DM energy density at \(a = 1\). Equations (6), (13), and (59) give the total rate of change of power law entropy on the apparent horizon:

\[
\dot{S}_{\text{tot}} = -\frac{2\pi^2}{H^3} \left( 1 + \frac{\dot{H}}{H^2} \right) \left( \frac{1}{T_m} - \frac{2\pi}{H} \right) (1 + \omega_m) \rho_0 a^{-4(1+\omega_m)}
\]

\[
\quad + \frac{3\pi^2 \dot{H}}{2H^2} \left[ \frac{\dot{H}}{H^2} + \frac{\lambda}{\pi^2 - \lambda} \left( \frac{\pi}{2H^3} \right)^{1-\frac{3}{2}} \right].
\]

In logarithmic corrected entropy, the GSLT on the apparent horizon leads to

\[
\dot{S}_{\text{tot}} = -\frac{2\pi^2}{H^3} \left( 1 + \frac{\dot{H}}{H^2} \right) \left( \frac{1}{T_m} - \frac{2\pi}{H} \right) (1 + \omega_m) \rho_0 a^{-4(1+\omega_m)}
\]

\[
\quad + \frac{3\pi^2 \dot{H}}{2H^2} \left[ \frac{\dot{H}}{H^3} - \frac{\mu H}{\pi^2} \right].
\]

Equations (60) and (61) imply that the GSLT is valid if \(T_m > \frac{\dot{H}}{2\pi}\), \(\frac{\dot{H}}{H^2} > -1\) along with the conditions mentioned after Eqs. (17) and (19), respectively. Further, we can obtain the total rate of change of power law entropy in the non-equilibrium condition on the event horizon in view of Eqs. (7), (13), and (59):

\[
\dot{S}_{\text{tot}} = -2\pi^2 R_e^3 (1 + \omega_m) \rho_0 a^{-4(1+\omega_m)} \left( \frac{1}{T_m} - 2\pi R_e \right)
\]

\[
\quad + \frac{3\pi^2 R_e^2}{2} \left[ \dot{H} R_e^2 + \dot{R}_e \left( 1 - \frac{\lambda}{\pi^2} \left( \frac{\pi R_e^3}{2} \right)^{1-\frac{3}{2}} \right) \right].
\]

Similarly, the GSLT for logarithmic corrected entropy in this scenario turns out to be

\[
\dot{S}_{\text{tot}} = -2\pi^2 R_e^3 (1 + \omega_m) \rho_0 a^{-4(1+\omega_m)} \left( \frac{1}{T_m} - 2\pi R_e \right)
\]

\[
\quad + \frac{3\pi^2 R_e^2}{2} \left[ \dot{H} R_e^2 + \dot{R}_e \left( 1 + \frac{2\mu}{\pi^2 R_e^3} \right) \right].
\]
The above two expressions of the GSLT indicate that it holds on the event horizon for $T_m > \frac{1}{2\pi R_e}$ with the other conditions given after Eqs. (20) and (21), respectively.

For the closed KK Universe, we have

$$\dot{S} = 2\pi^2 R_h^3 (\dot{R}_h - HR_h) \left[ 2\pi(1 + \omega_m)\rho_0 a^{-4(1+\omega_m)} \left( \frac{1}{T_m} - 2\pi R_h \right) - \frac{3\pi H^2 R_h}{2} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \right].$$  

(64)

The total rate of change of the power law form of entropy on the apparent horizon yields

$$\dot{S}_{tot} = -\frac{2\pi^2(1 + \omega_m)\rho_0}{H^3(1 + \Omega_k)^2} \left( 1 + \frac{\dot{H}}{H^2} - \Omega_k \right) \left( \frac{1}{T_m} - \frac{2\pi}{H\sqrt{1 + \Omega_k}} \right) a^{-4(1+\omega_m)}$$

$$+ \frac{3\pi^2}{2H^2(1 + \Omega_k)^2} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ \frac{\Omega}{r_e^2 - \lambda} \left( \frac{\pi}{2H^3(1 + \Omega_k)^2} \right)^{1 - \frac{\lambda}{2}} \right]$$

$$+ \frac{1}{1 + \Omega_k} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ \frac{H}{H^2} - \dot{\Omega}_k \right].$$  

(65)

For the logarithmic correction, the GSLT on the apparent horizon becomes

$$\dot{S}_{tot} = -\frac{2\pi^2(1 + \omega_m)\rho_0}{H^3(1 + \Omega_k)^2} \left( 1 + \frac{\dot{H}}{H^2} - \Omega_k \right) \left( \frac{1}{T_m} - \frac{2\pi}{H\sqrt{1 + \Omega_k}} \right) a^{-4(1+\omega_m)}$$

$$+ \frac{3\pi^2}{2H^2(1 + \Omega_k)^4} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ \left( \frac{H}{H^2} - \dot{\Omega}_k \right) \sqrt{1 + \Omega_k} - \frac{2\mu}{\pi^2 H^3} \right].$$  

(66)

We see from the above two expressions that the GSLT is valid on the apparent horizon for $T_m > \frac{H\sqrt{1 + \Omega_k}}{2\pi R_e}$ along with the conditions mentioned after Eqs. (39) and (42), respectively. However, on the event horizon, the GSLT takes the form

$$\dot{S}_{tot} = -2\pi^2 R_e^3 \left( \frac{1}{T_m} - 2\pi R_e \right) (1 + \omega_m)\rho_0 a^{-4(1+\omega_m)}$$

$$+ \frac{3\pi^2 R_e^2}{2} \left[ H^2 R_e^2 \left( \frac{H}{H^2} - \Omega_k \right) + \dot{R}_e \left( 1 - \frac{\Omega}{r_e^2 - \lambda} \left( \frac{\pi R_e^3}{2} \right)^{1 - \frac{\lambda}{2}} \right) \right].$$  

(67)

In the same entropy correction, the GSLT on the event horizon is

$$\dot{S}_{tot} = -2\pi^2 R_e^3 \left( \frac{1}{T_m} - 2\pi R_e \right) (1 + \omega_m)\rho_0 a^{-4(1+\omega_m)}$$

$$+ \frac{3\pi^2 R_e^2}{2} \left[ H^2 R_e^2 \left( \frac{H}{H^2} - \Omega_k \right) + \dot{R}_e \left( 1 + \frac{2\mu}{\pi^2 R_e^3} \right) \right].$$  

(68)
We would like to mention here that the GSLT given in Eqs. (67), (68) corresponding to the event horizon is respected for \( T_m > \frac{1}{2\pi R_c} \) along with the conditions mentioned after Eqs. (43) and (45).

VI. CONCLUDING REMARKS

In this paper, we have investigated the GSLT for a system consisting of interacting \( n \)-components of fluid and entropy corrected (power law and logarithmic) versions of the horizon entropy of the universe in a flat and closed KK universe. We have also considered thermal equilibrium and non-equilibrium conditions to discuss this law on the apparent as well as the event horizon in quintessence (\( \dot{H} < 0 \) and \( \dot{R}_c > 0 \)) and phantom (\( \dot{H} > 0 \) and \( \dot{R}_c < 0 \)) eras of the universe. The power law and logarithmic corrected entropy parameters such as \( \lambda \) and \( \mu \), respectively, play an important role in exploring the validity of the GSLT. In order to make more clear the justification of the validity of the GSLT, we have made plots of the GSLT versus \( t \) in the thermal equilibrium case by choosing the scale factor and some arbitrary values of the model parameters. In the flat KK universe with thermal equilibrium, we have explored the GSLT on the apparent horizon for small perturbations around the de Sitter spacetime and found all possible conditions for its validity for both entropy corrected versions.

Moreover, we have found different conditions for which the GSLT is valid on the event horizon in the quintessence and phantom regions in the flat KK universe. We have also used a phantom like scale factor and found its upper bounds for the validity of the GSLT. In the closed KK universe with thermal equilibrium, we have investigated the GSLT on the apparent as well as event horizon and have explored different possibilities for its validity generally. In particular, we have found conditions for the GSLT on the event horizon by using the phantom like scale factor. We have also evaluated the validity of the GSLT in the presence of DE and CDM components of the universe at the present epoch. We have chosen the well-known observational values of constant cosmological parameters and found that the GSLT holds in all cases in certain ranges of \( \lambda \) and \( \mu \).

We have also evaluated the GSLT for the thermal non-equilibrium case in the quintessence and phantom regions on the apparent and event horizons for the specific case of non-interacting DE and DM in a flat and a closed KK universe. We have determined general conditions for the validity of the GSLT in this scenario. We also remark that \( \dot{S}_{tot} \geq 0 \) under certain conditions for both entropy corrected versions in a KK universe either concerned with the thermal equilibrium or non-equilibrium of interacting components of fluid. These conditions have been developed in general without specifying any form of DE and the interaction term. Further, these are different (for flat KK universe) from those given in recent papers [25, 26] (for flat FRW universe) and more generally in a closed KK universe on the apparent and event horizons.
References


